COLLISION CALCULUS
BOUNDARY READINGS, ENERGY AND REFINEMENT

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Abstract. A finite sample records the value assigned at a boundary. An integral records the surrounding profile. When several components meet at that boundary, their signed amplitudes interact before the energy is measured. Collision calculus organizes these effects into a method for comparing continuous and sampled energy as fields, sampling locations and weights change. A line and three readings introduce the problem. A periodic boundary formula separates point contributions, jumps and curvature. Weighted summation then handles moving cutoffs and complete infinite tails. Linear combinations of readings that survive permitted additions give lower bounds on the energy those additions can leave behind.

Worked examples and independent problems develop the choice and use of these operations. A moving weighted signal has an explicit uniform energy estimate; two overlapping pulses have persistent defects of opposite signs on shifted grids. Arithmetic applications express the same data through congruences, divisor sums and clock fields. They include a finite prime-counting identity and a half-cell shift that changes the carry limit from one to one quarter. For tapered divisor fields, completed-cycle screening controls a specified range of interactions, while weighted localization encloses the full energy change by a finite sum and explicit tail allowances. Each calculation specifies the observation, retains the mixed terms and identifies the estimate required when the field grows.

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Author's Preface

I put this book together partly for myself. These are calculations I know well now, but I wanted something to come back to when I've been away from them long enough to get rusty. If you're interested in the methods, I hope having them in one place makes them easier to learn.

The material grew out of my work on digit collisions, carry boundaries and arithmetic refinement. I kept running into questions about how a continuous description relates to a finite table of readings. A value at one boundary can change a sampled energy without changing an integral. If several components meet there, their interaction has to be included as well. Following these effects as the field and its observation change is the common task behind the calculations here.

Collision calculus is the name I use for these boundary and energy calculations. Some of the methods are classical, including summation by parts and quadratic forms. Other constructions and estimates were developed in the course of the arithmetic work. I identify the sources and contributions as they enter the text; Section 1.8 gives an overview.

The book starts with a line and three sample points. Small examples let you see the conventions at work before the calculations get longer. I've included problems so there is something to work through along the way, with solutions at the back. The glossary and index are there for the next time either of us needs to look something up.

Reader's guide

If you're coming to this for the first time, I'd start with the line and three readings in Section 1. The calculation is small enough to do by hand, and it gives you something concrete to keep in mind when the notation arrives. Section 1.3 introduces unity and conserved mass, which recur when the fields and their observations change.

Calculus, finite sums and elementary linear algebra will get you through most of the first two parts. You can start with the real-valued examples and think of a finite positive measure as a list of nonnegative sample weights. If Fourier series are unfamiliar, you can use the hypotheses and formula of Theorem 4.1 in the examples, then come back to the proof.

Parts I and II are arranged so you can learn the methods before taking on the arithmetic applications. The moving window in Section 10 and the overlapping pulses in Section 11.2 bring several tools together. If you can work through those examples and account for each term, you have a useful basis for the later calculations.

The problems give you something to try as you go. The hints and solutions are at the back (page 103), and the answer numbers link back to the questions. You can use a hint to get started, or work through a solution and try the calculation again on your own.

The arithmetic work in Part 3 uses congruences and divisor sums, leading to the prime-counting identity. Part 4 applies the methods to growing carry and divisor fields. Some arguments there need the additional analytic estimates cited in the text. Section 1.8 gives the sources and identifies the constructions developed in this work.

If you're returning for a particular calculation, you can use the operation table in Section 7 to find a starting point. The symbol table in Section 3 and the glossary are there for the definitions. The index should get you back to the places where a tool is used.

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Part 1

Boundary readings and sampled energy

1. Continuous and sampled energy

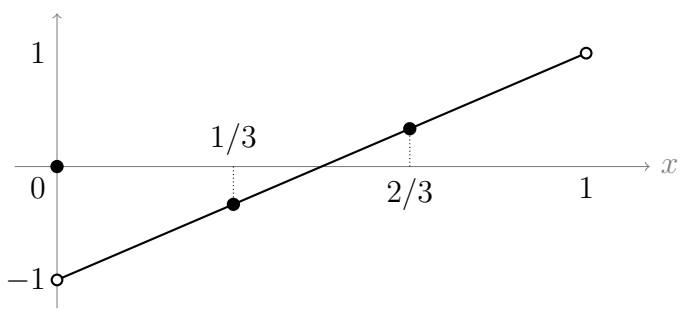
Suppose a periodic signal is known as a curve, but its energy is measured from a finite set of readings. An integral and a sample average are two ways of observing that signal. They need not give the same answer. This matters when a sampled energy is used to compare signals, choose a grid, or predict the effect of adding a new component. Before estimating their difference, we need to know which data each observation uses.

Here energy means squared magnitude, integrated or averaged. For a real reading u it is u^2; for a complex reading it is $|u|^2$. Squaring makes interaction unavoidable. Two contributions of size one give an energy of four when they reinforce and zero when they cancel. Their separate energies add to two in either case.

The first calculation needs only a straight line, three sample points and the sum of their squares. It will identify three questions that recur throughout the subject. Which value is recorded at a boundary? Which components meet there? What changes when the observation is refined?

1.1. **Reset values and sampled energy.** On $0 < x < 1$, take the line $f(x) = 2x - 1$ and repeat it with period one. The value approaches 1 before each integer and -1 after it. Assign the midpoint reading $f(0) = 0$, and the same reading at every integer. These assigned values are part of the field.

Sample at $0, 1/3, 2/3$. The curve and the three readings are shown below. Open circles mark the limiting values at the ends of the rising segment; filled circles mark actual samples.



The continuous energy and the sampled energy are

$$\int_0^1 (2x-1)^2\,dx = \frac{1}{3}, \qquad \frac{1}{3}\left(0^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = \frac{2}{27}.$$

Their difference, integral minus sample average, is $7/27$.

Now change only the reset reading to $g(0) = -1$. This is the right-hand reading of the same rising segments. Changing one point leaves the integral equal to $1/3$, but it increases the sample average by $1/3$. The two observations are

Reset reading	Continuous energy	Sampled energy	Difference
Midpoint 0	$1/3$	$2/27$	$7/27$
Right-hand -1	$1/3$	$11/27$	$-2/27$

The same open curve has produced discrepancies of opposite signs. The integral and sample average are both correct for their definitions. The reading convention determines which comparison is being made.

1.2. **Point contribution and signed remainder.** At the reset, the two limiting energies are both one. Their average is one, whereas the midpoint reading has energy zero. A sampled point of weight $1/3$ therefore contributes $1/3$ to the discrepancy. The whole discrepancy is $7/27$, leaving a signed contribution of $-2/27$ from the rest of the boundary formula.

The point contribution has been separated from the complete answer. For the right-hand reading, the point contribution vanishes and the same $-2/27$ remains. Later, the derivative jump will account for that term.

Problem 1 (Changing one of the three readings)**.** Keep the open curve above and set its reset reading to a real number c. Find its sampled energy and discrepancy. For which c is the discrepancy zero?

1.3. **Unity and refinement.** Unity is the constant field $\mathbf{1}(x) = 1$. It gives a fixed reference while a description becomes more detailed. Both observations in the opening example assign total mass one: the interval has length one, and the three sample weights add to one. Consequently both assign energy one to unity. Their disagreement about the sawtooth comes from its readings at those same addresses.

A whole can be resolved into unequal shares. The reciprocal weights of the later native arithmetic observation are

$$w_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \sum_{n=1}^{M} w_n = 1 - \frac{1}{M+1}.$$

The last term tends to zero, so all positive addresses together have mass one. Resolving the first two addresses leaves

$$1 = \underbrace{\frac{1}{2}}_{n=1} + \underbrace{\frac{1}{6}}_{n=2} + \underbrace{\frac{1}{3}}_{n\geq 3}.$$

Resolving the next address divides the last share into $1/12 + 1/4$. The remaining $1/4$ still includes infinitely many addresses. It is part of the same total at every stage.

This is *conservation of mass under subdivision*. If a cell of mass m is partitioned into children of masses $m_1, \ldots, m_k$, then

$$\sum_{j=1}^{k} m_j = m, \qquad \sum_{j=1}^{k} m_j |c|^2 = m|c|^2.$$

The second identity follows when each child inherits the same reading c. Finer resolution alone has preserved that field's energy. New child readings require a new energy calculation. The observation mass, the constant field unity and the energy of a general field are distinct objects. On an observation of mass one, the constant field two has energy four.

A resolved part and its complement. Let $u_M(n)$ be one on addresses $1, \ldots, M$ and zero on all later addresses. This indicator field represents unity exactly on the resolved part. Its residual $r_M = \mathbf{1} - u_M$ is one on the remaining tail. Their supports are disjoint, so

$$\sum_{n\geq 1} w_n |u_M(n)|^2 = 1 - \frac{1}{M+1}, \qquad \sum_{n\geq 1} w_n |r_M(n)|^2 = \frac{1}{M+1}.$$

For $M = 2$, the represented energy is $2/3$ and the error energy is $1/3$. Admitting the third indicator transfers $1/12$ from the error to the represented part. The target remains $\mathbf{1}$ throughout. Here the allowed fields are indicators; a prescribed clock family offers different corrections.

This fixed-target account motivates the refinement work in *Unity, Refinement, and Signed Capture* [6]. Section 9.1 develops its elementary clock example. Corrections there can overlap the old fit, so their signed interaction enters the error change. The mass partition identifies the whole being observed; the permitted fields determine how much of unity can be represented.

Problem 2 (Unity and unequal shares)**.** Split an observation of mass one into two addresses with masses $1/3, 2/3$. Give them real readings $1+a, 1+b$. Choose b in terms of a so that the weighted mean remains one. Compute the field's energy and its squared error relative to unity, including the case $a = 2$. Compare with the choice $b = -a$. Explain which quantities subdivision preserves and which depend on the child readings.

1.4. **Collision amplitudes and interaction.** Suppose a field has limiting values $u + v$ and $u - v$ on the two sides of a boundary, with the midpoint value u recorded at the boundary. The average of the limiting squared magnitudes is $|u|^2 + |v|^2$. The recorded square is $|u|^2$. Their local difference is $|v|^2$.

If several components reset together, their signed jumps add before this square is taken. This coincidence of boundary events is a *collision*. Its contribution depends on the combined reading, including the interaction between the resetting components.

For example, add a second copy of the three-sample field f. The combined field is $2f$, whose discrepancy is $4(7/27) = 28/27$. Adding the two individual discrepancies gives only $14/27$. The missing $14/27$ is the mixed contribution. If the second field is $-f$ instead, the combined field and its discrepancy vanish. The mixed contribution then cancels the two individual discrepancies.

Both calculations are consequences of

$$|f + g|^2 = |f|^2 + 2\,\mathrm{Re}(f\overline{g}) + |g|^2.$$

The mixed term must be observed using the same integral or sampling rule as the two squares, including when that rule changes.

1.5. **Grid refinement and field growth.** For the same midpoint field f, now sample at a/q, $0 \le a < q$. Elementary sums of integers and their squares give

$$\frac{1}{q}\sum_{a=1}^{q-1}(2a/q - 1)^2 = \frac{1}{3} - \frac{1}{q} + \frac{2}{3q^2}.$$

The discrepancy is therefore

$$D_q[f] = \frac{1}{q} - \frac{2}{3q^2} \longrightarrow 0. \tag{1}$$

Here D_q is simply a name for integral minus sampled mean square. The field was held fixed while the grid changed.

Now let the field grow with that grid: $f_q = \sqrt{q}\, f$. Both observed energies are multiplied by q, so the complete discrepancy is

$$D_q[f_q] = 1 - \frac{2}{3q} \longrightarrow 1. \tag{2}$$

This limit has been calculated directly, including its signed remainder. There is no conflict with the fixed-field limit. The changing amplitude has exactly offset the decreasing weight of one sample.

A collision of m equal components has squared amplitude proportional to m^2. On a q-point grid its point contribution can consequently have size m^2/q. This explains why the scale $m \asymp \sqrt{q}$ deserves attention. In a general growing family the remaining signed terms still require an estimate; formula (2) works because those terms were also computed.

Problem 3 (Choose the growth scale)**.** For real α, put $f_q = q^\alpha f$. Determine when the discrepancy tends to zero, tends to one, or grows without bound.

1.6. **Interval pulse.** Take a unit pulse supported on $[1/4, 3/4)$ and sample at $0, 1/4, 1/2, 3/4$. The continuous energy is the interval length $1/2$. Two of the four readings are one, so the sampled energy is also $1/2$. The discrepancy is zero.

Change the pulse to have value zero at both endpoints. The integral stays $1/2$, but only the sample at $1/2$ remains. The sampled energy is now $1/4$ and the discrepancy is $1/4$. This calculation uses the same point-replacement rule as the clock example. Rising segments and arithmetic divisibility are not required for that operation.

The later pulse calculation allows any interval, amplitude, endpoint readings and grid offset. It will show how a boundary formula replaces a fresh case-by-case count each time the sampling grid moves.

1.7. **Changes of field and observation.** The starting question is concrete: how does an observed squared norm change when its field or observation changes? The data needed for an answer depend on that change.

Change	Data that must be retained
Change a reading	The actual value and the weight of that sample.
Add a field	Its square and its interaction with the inherited field.
Move or refine a grid	Which boundaries are sampled and the new offset.
Grow the field	The dependence of every error constant on that field.
Change weights or activation	Both endpoints, weight variation and the complete tail.

These operations matter whenever an integral is used to compare or predict a finite squared observation. The examples will cover periodic signals, interval indicators, arithmetic carry fields and reciprocal address weights. Each application specifies its observation before applying the common rules.

1.8. **Classical foundations and specific contributions.** The general operations come from established analysis and linear algebra. Their classical names identify the mathematics being used.

Operation	Mathematical foundation
Boundary decomposition	Euler–Maclaurin summation with jumps and Bernoulli kernels [2, 3].
Field and observation change	Expansion and polarization of a squared norm; integration against atomic and continuous measures.
Weighted boundary work	Summation by parts, telescoping and convergence of weighted series.
Invariant readings	Weighted Cauchy–Schwarz and minimum-norm solutions under linear constraints [4, Sections 6.2 and 10.1.1].
Arithmetic registers and transforms	Congruence solving, gcd grouping and Möbius inversion [1, Chapters 2 and 5].

The point-value rule is the contribution of an atom to an integral. The positive midpoint term is the parallelogram identity. The localization rule

compares two nonnegative tail energies. Each is derived here in the form needed for the calculations; its elementary basis remains visible.

Unity in Section 1.3 is the usual constant function one. Its approximation uses weighted squared error, and its mass partitions use additivity of the observation. The arithmetic setting and the distinction between resolving a reading and admitting a new clock come from the refinement work [6].

The contribution of the general presentation is to put these operations on the same specified field and observation. A boundary reading, a combined reset, a moved sampling grid and a changed activation cutoff can then be followed through one calculation. The clock formula makes that organization explicit through the collision amplitude, inherited reading and slope. The moving-window and pulse examples show how to turn the identities into complete answers for growing families.

Sections 12–15 collect tools developed in the square-root sampling investigation. The signed register, finite-column estimate and interior norm construction retain that investigation's specific arithmetic data. Primitive-shell grouping and the logarithmic transform express classical divisor operations in a reusable form; their application to the clock remainder supplies the finite prime-counting identity. The derivations are included here.

The specialized results have a more particular provenance. The unshifted carry limit comes from *The Unit Sampling Defect in Digit-Collision Energy* [5]. The half-cell source reduction, parity transfer and quarter-limit are developed in Section 17, using analytic estimates from that work. They are the principal new arithmetic development of this manuscript.

The native screening constant 129, cycle-alignment credit and parent-shell constructions were developed in the divisor-field investigation. Sections 18–20 derive them from the prescribed family profiles and include their proofs. The quadratic cutoff is an explicit application of the general localization rule. The four-reading construction specializes constrained energy to the later divisor families. Its failed sufficiency comparison at the four stated cutoffs is an exact result of the present investigation. These are specific formulas, constructions and estimates, with their derivations and dependencies identified. Collision calculus names this coordinated method of boundary and energy accounting.

For the next problem, a grid with q points and offset θ samples $(a+\theta)/q$, $0 \le a < q$. The offset is measured in cells: $\theta = 1/2$ moves each sample by half a cell, or $1/(2q)$.

Problem 4 (An isolated reading)**.** Let u vanish on the periodic interval except for $u(0) = 2$. Compute its continuous energy and its sampled energy on the four-point grids with offsets $\theta = 0$ and $\theta = 1/2$. Determine both defects. Explain which datum makes the answers different.

2. Fields and observations

A field supplies readings. An observation assigns mass to the addresses at which those readings are taken. Here an address is simply a location in the domain, such as a point of an interval or an integer index. Keeping both objects visible lets the same calculation describe a grid, an interval integral, a moving window or an infinite weighted tail.

A two-address example separates the two objects. Give the readings $1, 3$ masses $1/2, 1/2$. The observed energy is $(1^2 + 3^2)/2 = 5$. Keep the readings but change the masses to $3/4, 1/4$; the energy becomes 3. Keep those new masses and change the second reading to 1; the energy becomes 1. The first change belongs to the observation, the second to the field. We need notation that records both.

For a finite table, the definitions below reduce to these weighted sums. A measure records the mass assigned to each measurable set of addresses. Its integral adds readings with those masses, extending the same operation to a continuous profile or an infinite weighted tail.

Let X be a measurable address space whose singletons are measurable. An observation ρ is a finite positive measure on X; its mass need not be one. A field is a specified measurable function on X, including its readings at atoms of ρ. Define

$$Q_\rho h = \int_X h \, d\rho, \qquad E_\rho[f] = Q_\rho(|f|^2), \qquad K_\rho(f, g) = Q_\rho(f\overline{g}).$$

We use Q_ρ only when the integral exists and use E_ρ and K_ρ on fields of finite energy. Cauchy–Schwarz then gives $|K_\rho(f, g)|^2 \leq E_\rho[f]E_\rho[g]$. The ordinary letters describe the operation; subscripts describe the observation. These are the usual integral, squared norm and mixed form for the chosen measure. Their value here is that the observation stays explicit when its addresses or masses change.

When two observations ρ, σ are compared, write

$$D_{\rho,\sigma}[f] = E_\rho[f] - E_\sigma[f], \qquad K_{\rho,\sigma}(f, g) = K_\rho(f, g) - K_\sigma(f, g).$$

Both fields must have finite energy in both observations. The difference D can have either sign. It is quadratic in the field, and its mixed form $K_{\rho,\sigma}$ need not be positive.

Observation	Measure on its address space
Periodic integral	dx on $\mathbb{T}$.
Shifted grid	$q^{-1}\sum_{a=0}^{q-1}\delta_{(a+\theta)/q}$.
Weighted window	$\sum_{a<n\le b} v_n\delta_n$, with $v_n \ge 0$.
Native tail	$\sum_{n>S}[n(n+1)]^{-1}\delta_n$.

Here δ_x assigns mass one to the single address x. Thus an entry $v\delta_x$ in the observation means "read at x with weight v." A collection of such point masses turns the integral into the corresponding weighted sum.

In the grid setting the first comparison measure is understood to be dx, so $D_{q,\theta}$ and $K_{q,\theta}$ abbreviate these two-measure operations. Later E_S abbreviates the native tail. Normalizing a tail to mass one would define a different observation and change its energy.

2.1. **Energy under changes of field and observation.** Suppose the old experiment observes f with ρ, and the new one observes $f+g$ with σ. The field f must be specified at every new address, even where the old observation assigned no mass.

Proposition 2.1 (Change of field and observation)**.** *If f has finite energy under ρ and σ, and g has finite energy under σ, then*

$$E_\sigma[f+g] - E_\rho[f] = -D_{\rho,\sigma}[f] + 2\,\mathrm{Re}\,K_\sigma(f,g) + E_\sigma[g]. \tag{3}$$

For a common atomic address space with old masses u_n and new masses v_n, the first term is $\sum_n (v_n - u_n)|f(n)|^2$. It includes every removed address, newly admitted address and change of retained mass.

Proof. Expand $|f+g|^2 = |f|^2 + 2\,\mathrm{Re}(f\overline{g}) + |g|^2$ under σ and subtract $E_\rho[f]$. The two square integrals converge by hypothesis, and Cauchy–Schwarz makes the mixed integral absolutely convergent. In the atomic case $\sum_n |v_n - u_n||f(n)|^2 \le E_\sigma[f] + E_\rho[f]$, so its series also converges absolutely. This proves the formula without an unjustified rearrangement of an infinite sum. □

The three terms answer different parts of the calculation. The first charges the old field for changing its observation. The second measures

the new field's signed interaction with what was already present. The third is the new field's own energy.

For a small calculation, give addresses $1, 2, 3$ old masses $(1/2, 1/3, 1/6)$ and new masses $(0, 1/2, 1/2)$. Take $f = (1, 2, 0)$ and $g = (0, -1, 1)$. The energies are $11/6$ and 1. The three terms on the right are respectively $1/6, -2, 1$. Their sum is $-5/6$. Removing address one does not determine the answer by itself; reweighting address two and its negative mixed term also contribute.

2.2. **Readings, superposition and consistency.** If two fields differ only on an at most countable set A, have finite energy, and agree elsewhere, then

$$E_\rho[g] - E_\rho[f] = \sum_{a \in A} \rho(\{a\}) \big(|g(a)|^2 - |f(a)|^2 \big).$$

This is the general point-value replacement rule. Its series is absolutely convergent. An atomless observation ignores the changed readings; an atomic observation charges their actual masses.

For finitely many components $f = \sum_j c_j f_j$, the matrix $G_{ij} = K_\rho(f_j, f_i)$ gives $E_\rho[f] = c^* G c$. Here a collision means that components contribute at a common observed address; the mixed term measures their interaction. Components with disjoint observed support have zero mixed energy. Components meeting at a boundary contribute through their combined reading. The interaction may reinforce or cancel in the combined energy.

Changing an experiment in two stages gives a useful consistency check. Apply (3) to each stage and add. The intermediate energy cancels, leaving the direct change. The allocation among the three terms can depend on the chosen intermediate field and observation; their total cannot. This check catches a lost shell or a duplicated boundary credit before any estimate is attempted.

Problem 5 (Changing masses and readings)**.** On three addresses take old masses $(1/2, 1/2, 0)$, new masses $(0, 1/3, 2/3)$, old field $f = (1, -1, 2)$ and change $g = (0, 1, -1)$. Calculate the energy change directly. Then recover it as the sum of observation change, inherited interaction and new square. Keep the old field's reading at the newly admitted address.

3. Fields, traces and sampled energy

The first example distinguished a curve from its assigned readings. We now record that distinction for any periodic field with finitely many boundaries. Work on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the unit interval with its ends identified. Write $\lfloor x \rfloor$ for the greatest integer at most x, $\lceil x \rceil$ for the least integer at least x, and $\{x\} = x - \lfloor x \rfloor$ for the fractional part. The notation $\mathbf{1}_A$ is one on the set A and zero elsewhere.

An admissible field $f : \mathbb{T} \to \mathbb{C}$ has a finite set $\mathcal{B}$ of specified boundary points. On each open arc between consecutive points of $\mathcal{B}$, it extends to a C^2 function on the closed arc. The extensions from adjacent arcs may differ. The actual value $f(b)$ is specified separately at every $b \in \mathcal{B}$. Include isolated exceptional readings in $\mathcal{B}$, even if the two traces agree. A smooth field may be given an artificial boundary at zero.

The C^2 condition means that the function and its first two derivatives are continuous on each closed piece. It permits integration by parts inside each piece while keeping its endpoint terms. A trace is the limiting value reached from one side. In the opening sawtooth, the left trace at zero is 1, the right trace is -1, and the assigned reading is 0. These are three distinct pieces of data.

For any function h with these properties, write

$$[h]_b = h(b+) - h(b-), \qquad \langle h \rangle_b = \frac{h(b+) + h(b-)}{2}, \qquad a_b(h) = \langle h \rangle_b - h(b).$$

All traces at zero are periodic traces. The notation h''_{reg} means the ordinary second derivative on the open arcs, ignoring the finitely many boundary points. The point data are not identified modulo sets of measure zero.

For an integer $q \geq 1$ and $0 \leq \theta < 1$, define

$$X_{q,\theta} = \left\{ \frac{a+\theta}{q} : 0 \leq a < q \right\}, \qquad Q_{q,\theta} h = \frac{1}{q} \sum_{a=0}^{q-1} h\left(\frac{a+\theta}{q} \right).$$

The linear sampling discrepancy and the quadratic energy defect are

$$L_{q,\theta}(h) = \int_0^1 h \, dx - Q_{q,\theta} h, \qquad D_{q,\theta}[f] = L_{q,\theta}(|f|^2).$$

We abbreviate $Q_{q,0}$ and $D_{q,0}$ to Q_q and D_q. The square is taken before sampling. Define the periodic kernels

$$\psi(t) = \begin{cases} \{t\} - 1/2, & t \notin \mathbb{Z}, \\ 0, & t \in \mathbb{Z}, \end{cases} \qquad \widetilde{B}_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}.$$

The first kernel takes its midpoint value at a jump. The second is continuous. Their respective absolute bounds are $1/2$ and $1/6$. Ordinary letters Q, D, K denote sampling, energy defect and mixed defect. The offset may be suppressed only after its value has been fixed. The kernels will record where a boundary falls within a sample cell. For a boundary b, the number $qb - \theta$ is an integer precisely when the boundary is sampled. Its fractional part records the relative position otherwise. Thus the apparently global sampling problem has local phase data at each boundary.

3.1. **Boundary data.** The periodic integral identifies functions that differ at finitely many points. A finite sampling rule need not do so. For example, let f vanish everywhere except at zero, where $f(0) = a$. Then

$$\int_0^1 |f|^2 \, dx = 0, \qquad D_q[f] = -\frac{|a|^2}{q}.$$

Both traces at zero vanish. The entire defect comes from the specified point value. This field is admissible after zero is included in $\mathcal{B}$. If the grid misses zero, its defect is zero as well.

The two measures being compared are Lebesgue measure and the atomic measure $q^{-1} \sum_{x \in X_{q,\theta}} \delta_x$. Their difference is a signed measure. A positive defect means the continuous energy exceeds the sampled energy; a negative defect means the sampled energy exceeds the continuous energy. Neither sign is prescribed in advance.

Symbol	Data or operation
q, θ	Number of samples and offset measured in grid cells.
$\mathcal{B}$	Finite set containing all jumps and exceptional readings.
$[h]_b$	Right trace minus left trace at a boundary.
$a_b(h)$	Average trace minus the actual point reading.
$L_{q,\theta}$	Linear difference of integral and sample average.
$D_{q,\theta}$	The same difference applied to squared magnitude.
$K_{q,\theta}$	Mixed energy difference for two fields.
$P_{q,\theta}, R_{q,\theta}$	Sampled point contribution and the remaining signed defect.
E_S	Native weighted energy on addresses strictly greater than S.
N	Local size parameter: a cutoff in weighted sums, a scale in the moving-window model, and the clock count $p-1$ in the carry application.

3.2. **Point-value replacement.**

Proposition 3.1 (Point-value replacement)**.** *Suppose admissible fields f, g agree outside a finite set S. Then*

$$D_{q,\theta}[g] - D_{q,\theta}[f] = \frac{1}{q} \sum_{b \in S \cap X_{q,\theta}} \big(|f(b)|^2 - |g(b)|^2\big).$$

Proof. The squared integrals agree. In the sample sums, only the readings in $S \cap X_{q,\theta}$ can differ. Subtracting those sums gives the result. □

This identity permits a calculation to use a convenient trace convention and then restore the required readings exactly. The correction depends on the grid: changing an unsampled point has no effect.

Problem 6 (Boundary readings and traces)**.** On the periodic interval let $f = 1$ on $(0, 1/2)$ and $f = 3$ on $(1/2, 1)$, with $f(0) = f(1/2) = 2$. Record the two traces and the actual reading of f and $h = f^2$ at both boundaries. Find $a_b(h)$ and $[h]_b$. Compute $Q_q(h)$, $D_q[f]$ and the sampled point contribution for $q = 3, 4$. Compare $Q_q(f^2)$ with $(Q_q f)^2$.

4. Boundary decomposition

The three-sample calculation had a point contribution and a signed remainder. The following formula separates that remainder into value jumps, slope jumps and curvature between boundaries. Its four terms can be calculated from local data before any asymptotic estimate is made.

Euler–Maclaurin summation supplies the underlying relation between sums, integrals and boundary data. Its circular formulation treats interior and periodic jumps [2]; shifted sampling with jumps is developed by Berrut and Trummer [3]. The Bernoulli kernels used below are defined in Section 3 and their Fourier expansions enter the proof. The purpose here is to teach how actual readings and quadratic interactions enter that relation, and how to preserve them through changes of observation.

For use of the formula, there are four entries to prepare. At each boundary record the actual energy and its two traces, then the two slope traces. Between boundaries retain the ordinary second derivative. The grid supplies the phase $qb-\theta$ at each boundary. The theorem weights those entries by two fixed periodic functions. Its first term records only boundaries actually sampled; the remaining terms account for the surrounding profile. The proof derives those weights from Fourier series, but the calculation itself uses the listed local data.

The proof uses a different description of the same data. Integration by parts records the jumps in each Fourier coefficient. Averaging a Fourier mode over the grid removes it unless its frequency is a multiple of q. Summing the surviving modes produces the two Bernoulli kernels. The point replacement at the end restores the actual boundary readings. This is why the formula needs both the surrounding traces and the specified values.

Theorem 4.1 (Boundary decomposition)**.** *Let $h:\mathbb{T}\to\mathbb{C}$ be piecewise C^2 with specified point values as above. For every $q\geq 1$ and $0\leq\theta<1$,*

$$L_{q,\theta}(h)=\frac{1}{q}\sum_{b\in\mathcal{B}\cap X_{q,\theta}}a_b(h)-\frac{1}{q}\sum_{b\in\mathcal{B}}[h]_b\psi(qb-\theta) \tag{4}$$
$$+\frac{1}{2q^2}\sum_{b\in\mathcal{B}}[h']_b\widetilde{B}_2(qb-\theta)+\frac{1}{2q^2}\int_0^1 h''_{\mathrm{reg}}(x)\widetilde{B}_2(qx-\theta)\,dx.$$

In particular, this formula with $h=|f|^2$ gives the complete energy defect of every admissible field. Boundary coincidences with the grid are included without a limiting convention for the sampled value.

Proof. First replace $h(b)$ by $\langle h\rangle_b$ at each boundary, and call the result h_*. The integral does not change, and

$$L_{q,\theta}(h) = L_{q,\theta}(h_*) + \frac{1}{q}\sum_{b\in\mathcal{B}\cap X_{q,\theta}} a_b(h). \tag{5}$$

Put $\widehat{h}(n) = \int_0^1 h(x)e^{-2\pi inx}\,dx$. Integration by parts twice on every arc, with the periodic endpoint terms retained, gives for $n \neq 0$

$$(2\pi in)^2\widehat{h}(n) = \widehat{h''_{\mathrm{reg}}}(n) + \sum_b [h']_b e^{-2\pi inb} + 2\pi in\sum_b [h]_b e^{-2\pi inb}. \tag{6}$$

Equivalently, the distributional second derivative has the regular part h''_{reg} and the point terms $\sum_b [h']_b\delta_b + \sum_b [h]_b\delta'_b$.

For $0 < \rho < 1$, the Abel means $h_\rho(x) = \sum_n \rho^{|n|}\widehat{h}(n)e^{2\pi inx}$ converge to $h_*(x)$ at every point. Indeed, the Poisson kernel is even and has total mass one; its mass outside any neighborhood of zero tends to zero. Its two half-neighborhoods consequently select the average of the two traces. Sampling the absolutely convergent series and using finite geometric sums yields

$$Q_{q,\theta}h_\rho - \int_0^1 h\,dx = \sum_{\ell\neq 0}\rho^{q|\ell|}\widehat{h}(q\ell)e^{2\pi i\ell\theta}.$$

Insert (6). The kernel identities are

$$\lim_{r\uparrow 1}\sum_{\ell\neq 0}\frac{r^{|\ell|}e^{-2\pi i\ell t}}{2\pi i\ell} = \psi(t),$$

$$\sum_{\ell\neq 0}\frac{e^{-2\pi i\ell t}}{(2\pi i\ell)^2} = -\frac{1}{2}\widetilde{B}_2(t).$$

They follow from the Fourier coefficients of the two Bernoulli functions; the first has value zero at integers by pairing opposite frequencies. The second series converges absolutely and uniformly. The regular integral may therefore be interchanged with its series and limit, since h''_{reg} is integrable. The boundary sums are finite. Taking $\rho \uparrow 1$ proves (4) for h_*; (5) restores the prescribed readings. □

Corollary 4.2 (Midpoint reading)**.** *If $f(b) = (f(b+) + f(b-))/2$, then*

$$a_b(|f|^2) = \frac{1}{4}|[f]_b|^2.$$

Proof. Write $u = (f(b+) + f(b-))/2$ and $v = (f(b-) - f(b+))/2$. The identity $(|u+v|^2 + |u-v|^2)/2 = |u|^2 + |v|^2$ gives the claim. □

The midpoint of a field and the midpoint of its energy are different readings. At an integer-zero sawtooth reset the field takes its midpoint, so its energy is smaller than the midpoint of the two energy traces. The resulting point correction is positive. The signed terms in (4) need not be positive.

Corollary 4.3 (A sufficient residual estimate)**.** *Set*

$$A_{q,\theta}(h) = \frac{1}{q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} a_b(h).$$

For every constant $c \in \mathbb{C}$,

$$|L_{q,\theta}(h) - A_{q,\theta}(h)| \le \frac{1}{2q} \sum_b |[h]_b| + \frac{1}{12q^2} \left(\sum_b |[h']_b| + \int_0^1 |h''_{\mathrm{reg}}(x) - c| \, dx \right).$$

For a sequence h_ν, q_ν, θ_ν, if the right side tends to zero and $A_{q_\nu,\theta_\nu}(h_\nu) \to A$, then $L_{q_\nu,\theta_\nu}(h_\nu) \to A$.

Proof. The integral of $\widetilde{B}_2(qx - \theta)$ over $\mathbb{T}$ is zero. Subtract c in the regular integral and use the two kernel bounds. □

This is an absolute sufficient condition. A growing arithmetic family may require cancellation in the signed sums when their absolute bounds do not decay. The decomposition identifies those sums without supplying an estimate for them.

4.1. **Polynomial curvature.** Let $f(x) = x^2$ on $0 \le x < 1$ and extend it periodically, retaining $f(0) = 0$. Its energy profile is $h(x) = x^4$ on that interval. At the periodic join, the boundary data are

$$a_0(h) = \frac{1}{2}, \qquad [h]_0 = -1, \qquad [h']_0 = -4, \qquad h''_{\mathrm{reg}}(x) = 12x^2.$$

On the unshifted q-grid the value-jump term vanishes because $\psi(0) = 0$. The point and derivative terms are $1/(2q)$ and $-1/(3q^2)$. To calculate the regular term, split the integral into the q cells and set $qx = a + t$ in cell a. Since

$$\int_0^1 \widetilde{B}_2(t) \, dt = \int_0^1 t\widetilde{B}_2(t) \, dt = 0, \qquad \int_0^1 t^2 \widetilde{B}_2(t) \, dt = \frac{1}{180},$$

direct expansion gives

$$\int_0^1 x^2 \widetilde{B}_2(qx)\,dx = \frac{1}{q^3}\sum_{a=0}^{q-1}\int_0^1 (a+t)^2 \widetilde{B}_2(t)\,dt = \frac{1}{180q^2}.$$

Thus the complete answer is

$$D_q[f] = \frac{1}{2q} - \frac{1}{3q^2} + \frac{1}{30q^4}.$$

For $q = 2$ it is $27/160$. Directly, the squared integral is $1/5$ and the two sampled squares have mean $1/32$, giving the same result. The q^{-4} contribution would disappear if regular curvature were discarded. This example follows the full procedure from the specified reading to the boundary data, the regular integral and a direct check.

Problem 7 (A periodic ramp)**.** Let $f(x) = x$ on $0 \le x < 1$ and repeat it periodically, with $f(0) = 0$. Find $D_q[f]$ for every $q \ge 1$ in two ways. First use a sum of squares; then list the point correction, value jump, derivative jump and regular curvature of $|f|^2$. Explain why the regular integral vanishes even though that curvature is nonzero.

Problem 8 (Smooth resonance)**.** For an integer $m \ge 1$, take $f_m(x) = \sin(2\pi m x)$. Compute its continuous energy and its sampled energies on the unshifted grids $q = 2m$ and $q = 2m + 1$. Identify the term of the boundary formula that accounts for any nonzero defect. For fixed m, determine the defect whenever $q > 2m$. Explain the distinction between that statement and the sequence of fields sampled at $q = 2m$.

5. Simultaneous arithmetic resets

For clocks, the local data have a particularly simple form. Each reset has a known jump, and each open segment has a constant slope. Simultaneous resets must first be grouped into a single signed amplitude. The clock $2\psi(2x)$ repeats the opening ramp twice during one trip around the circle, resetting at zero and one half. In a general clock $2c\psi(rx+\phi)$, r counts repetitions, ϕ chooses the phase, and c sets the signed or complex amplitude. The three parameters change different parts of the boundary calculation.

Let $r_1, \dots, r_s$ be positive integers, let $\phi_1, \dots, \phi_s \in \mathbb{T}$, and let $c_1, \dots, c_s \in \mathbb{C}$. Set

$$F(x) = 2\sum_{j=1}^{s} c_j \psi(r_j x + \phi_j), \qquad V = 2\sum_{j=1}^{s} c_j r_j.$$

Let $\mathcal{B}$ be the union of all reset locations. At $b \in \mathcal{B}$ define the collision set and its amplitude by

$$I_b = \{j : r_j b + \phi_j \in \mathbb{Z}\}, \qquad C_b = \sum_{j \in I_b} c_j.$$

The nonreset reading is $U_b = 2\sum_{j \notin I_b} c_j \psi(r_j b + \phi_j)$. Every location is counted once, with all its resetting clocks included in I_b.

Theorem 5.1 (Clock collision formula)**.** *For all such clocks, every integer $q \geq 1$ and every $0 \leq \theta < 1$,*

$$D_{q,\theta}[F] = \frac{1}{q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} |C_b|^2 + \frac{4}{q} \sum_{b \in \mathcal{B}} \mathrm{Re}(U_b \overline{C_b}) \psi(qb - \theta) \tag{7}$$
$$- \frac{2}{q^2} \sum_{b \in \mathcal{B}} \mathrm{Re}(V \overline{C_b}) \widetilde{B}_2(qb - \theta).$$

No coprimality or distinct-frequency hypothesis is required.

Proof. At each boundary,

$$F(b-) = U_b + C_b, \quad F(b+) = U_b - C_b, \quad F(b) = U_b.$$

For $h = |F|^2$, these give

$$a_b(h) = |C_b|^2, \qquad [h]_b = -4\,\mathrm{Re}(U_b \overline{C_b}), \qquad [h']_b = -4\,\mathrm{Re}(V \overline{C_b}).$$

On every open arc $F' = V$ and $F'' = 0$, so $h''_{\mathrm{reg}} = 2|V|^2$ is the same constant on every arc. Its Bernoulli integral vanishes. Theorem 4.1 now gives (7). □

For unit coefficients, a collision of m clocks contributes m^2/q if the grid samples it. With general coefficients,

$$|C_b|^2 = \sum_{j \in I_b} |c_j|^2 + 2\,\mathrm{Re} \sum_{\substack{j<k \\ j,k \in I_b}} c_j \overline{c_k}.$$

The reset budget already contains interference. Two clocks with the same frequency and phase and coefficients $1, -1$ have zero total field and zero collision amplitude at every reset. A count of individual resets alone would miss this cancellation.

5.1. **Two resetting clocks.** For $F = 2\psi(x) + 2\psi(2x)$, the slope on every open segment is $V = 6$. The distinct boundary locations are zero and $1/2$. Their data are

b	C_b	U_b	V
0	2	0	6
1/2	1	0	6

Thus the interaction term vanishes at both locations. On the unshifted nine-point grid, only zero is sampled, and

$$\sum_b C_b \widetilde{B}_2(9b) = 2\widetilde{B}_2(0) + \widetilde{B}_2(1/2) = \frac{1}{4}.$$

The clock formula gives

$$D_9[F] = \frac{4}{9} - \frac{12}{81}\frac{1}{4} = \frac{11}{27}.$$

On the half-cell grid, only $1/2$ is sampled. Now

$$\sum_b C_b \widetilde{B}_2(9b - 1/2) = 2\widetilde{B}_2(1/2) + \widetilde{B}_2(0) = 0,$$

and $D_{9,1/2}[F] = 1/9$. The continuous field has not changed. The two sampling rules select different collision amplitudes.

Phase shift and inherited interaction. The zero interaction above comes from its particular reset locations. Shift the second clock and take

$$F(x) = 2\psi(x) + 2\psi(2x + 1/4).$$

The slope is still $V = 6$, but the nonreset readings now contribute. The complete boundary data are

b	C_b	U_b	$F(b-)$	$F(b+)$
0	1	$-1/2$	$1/2$	$-3/2$
3/8	1	$-1/4$	$3/4$	$-5/4$
7/8	1	$3/4$	$7/4$	$-1/4$

At every reset the actual reading is $F(b) = U_b$. On the unshifted two-point grid, only zero is a sampled reset. Also $\psi(3/4) = 1/4$ and $\widetilde{B}_2(3/4) = -1/48$, so the three terms of (7) are

$$\underbrace{\frac{1}{2}}_{\text{sampled reset}} + \underbrace{2\left(-\frac{1}{4}\frac{1}{4} + \frac{3}{4}\frac{1}{4}\right)}_{\text{inherited interaction}} - \underbrace{3\left(\frac{1}{6} - \frac{1}{48} - \frac{1}{48}\right)}_{\text{slope contribution}} = \frac{1}{2} + \frac{1}{4} - \frac{3}{8} = \frac{3}{8}.$$

The actual samples are $F(0) = F(1/2) = -1/2$, giving sampled energy $1/4$. For an independent integral check, the three open segments are

$$F(x) = \begin{cases} 6x - 3/2, & 0 < x < 3/8, \\ 6x - 7/2, & 3/8 < x < 7/8, \\ 6x - 11/2, & 7/8 < x < 1. \end{cases}$$

Integrating their squares gives $27/128 + 13/32 + 1/128 = 5/8$. Thus the defect is $5/8 - 1/4 = 3/8$, as the boundary formula gives. Dropping the inherited term would give $1/8$. Its contribution comes from the two unsampled boundaries, through the nonreset readings there.

5.2. **Clock boundary calculation.** For a clock family with specified coefficients, list distinct reset locations before taking any squares. At each location, sum the coefficients of the resetting clocks to obtain C_b, and evaluate only the nonreset clocks to obtain U_b. The global slope V is computed once. A boundary is sampled exactly when $qb - \theta$ is an integer. These data supply all three terms of (7).

This procedure also provides a useful diagnostic. If the amplitudes of two resetting clocks cancel, their joint point correction disappears, but their interactions at other boundaries must still be evaluated. Cancellation at one location does not remove either clock from the field.

5.3. **Reset-reading problem.**

Problem 9 (A different reset reading)**.** Let $f = 2\psi$ and let $g = f$ away from zero, with $g(0) = 1$. Compute $D_q[g]$ and explain its sign.

Problem 10 (Opposing resets)**.** Take two clocks with opposite coefficients, $f(x) = 2\psi(x)$ and $g(x) = -2\psi(2x)$, using the specified midpoint readings. Put $F = f + g$. List the component reset locations and their combined collision amplitudes. Compute $D_4[F]$ directly, then compare it with $D_4[f] + D_4[g]$. Determine the mixed contribution and explain the effect of the opposing coefficients at zero.

6. Interval pulses and endpoint conventions

A pulse asks a counting question. How many of the sample addresses fall inside its interval, and are the endpoints admitted? Moving an endpoint changes the integral continuously, but a sampled reading changes when the

endpoint passes it. The boundary formula describes both effects using the same two endpoint locations.

The same theorem applies to a field with two boundaries and no rising clock segments. Fix $0 < \alpha < \beta < 1$ and $A \in \mathbb{C}$. Let $f = A$ on (α, β) and $f = 0$ outside $[\alpha, \beta]$. Specify $f(\alpha) = u$ and $f(\beta) = v$ independently.

Proposition 6.1 (Pulse boundary identity)**.** *For every q, θ,*

$$D_{q,\theta}[f] = \frac{|A|^2/2 - |u|^2}{q}\mathbf{1}_{\{q\alpha-\theta\in\mathbb{Z}\}} + \frac{|A|^2/2 - |v|^2}{q}\mathbf{1}_{\{q\beta-\theta\in\mathbb{Z}\}} + \frac{|A|^2}{q}\big(\psi(q\beta-\theta) - \psi(q\alpha-\theta)\big). \tag{8}$$

For the half-open convention $u = A, v = 0$, this also equals

$$|A|^2\left(\beta - \alpha - \frac{\lceil q\beta-\theta\rceil - \lceil q\alpha-\theta\rceil}{q}\right). \tag{9}$$

Proof. The energy jumps by $|A|^2$ at α and by $-|A|^2$ at β. All regular derivatives and derivative jumps vanish. Theorem 4.1 gives (8). Under the half-open convention the grid points in the pulse correspond exactly to integers a with $q\alpha - \theta \le a < q\beta - \theta$. Their number is the ceiling difference in (9). □

For a fixed amplitude and fixed endpoint readings, the defect is $O(q^{-1})$ uniformly in the grid offset. A persistent defect can be produced by growing the amplitude. Let q tend to infinity through multiples of three, take $\alpha = 1/3$, $\beta = 2/3$, $\theta = 0$ and $A_q = \sqrt{\lambda q}$ with $\lambda \ge 0$. Both endpoints lie on the grid. The four conventions below have identical continuous energy.

Endpoint readings	Interpretation	$D_q[f_q]$
$(A_q, 0)$	half-open pulse	0
$(0, 0)$	both boundaries omitted	λ
$(A_q/2, A_q/2)$	midpoint field readings	$\lambda/2$
(A_q, A_q)	both boundaries included	$-\lambda$

These values follow directly from (8). They show why a constant defect belongs to a specified family and reading convention. No universal nonzero constant is forced by discreteness alone.

Overlapping pulses also retain their interactions. For half-open intervals I_j, let $f = \sum_j A_j \mathbf{1}_{I_j}$. Then the energy contains $2\,\mathrm{Re}(A_j\overline{A_k})\mathbf{1}_{I_j\cap I_k}$ for each

pair $j < k$. Formula (11) keeps these overlap contributions. Their boundaries may coincide even when the intervals themselves differ.

6.1. **Overlap problem.**

Problem 11 (The cost of dropping overlap)**.** Let $I = [1/4, 3/4)$, $J = [1/2, 1)$ on $\mathbb{T}$, and $f = \mathbf{1}_I + \mathbf{1}_J$. Compute $D_5[f]$ and compare it with the sum of the two individual defects.

Problem 12 (Two adjacent pulses)**.** Put $f = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$ on the periodic interval. For an even integer q, calculate $D_q[f]$. Change only the combined field's readings at 0 and 1/2 to zero, obtaining g. Calculate $D_q[g]$ and identify the complete contribution of the replacement.

Problem 13 (Two refinement sequences)**.** Take $f = \mathbf{1}_{[0,1/3)}$ and $f_q = \sqrt{q}\, f$ on the unshifted grid. Calculate both defects when $q = 3m$ and when $q = 3m + 1$. Does $D_q[f]$ tend to zero along both sequences? Does $D_q[f_q]$ have a common limit? Decide from the count, without a fixed-field error constant.

Part 2

Field and observation changes

7. Boundary calculations

The choice of operation depends on the data available. Write the observed energies, then choose the formula whose required data can be evaluated or bounded.

7.1. **Readings and changed data.** Give the addresses, their masses and the actual readings. A native tail beginning after S starts at $S+1$. Write a changed field as $f+g$ on the union of old and new addresses. The following choices determine which additional data the calculation needs.

Task	First operation	Required data
Compare integral and grid	Boundary formula, Theorem 4.1	Traces, jumps, regular derivatives and offset.
Bound weighted interaction	Summation by parts, (10)	Primitive bound and variation of the weighted partner.
Change field and observation	Energy-change identity, Proposition 2.1	Old and new masses, inherited field and increment.
Restrict additions	Invariant readings, Theorem 8.3	Allowed additions and a preserved linear reading.

An energy change may first be decomposed and its mixed term then bounded by summation by parts. The weighted calculation follows.

7.2. **Observation weights and primitive sums.** Cancellation can be visible in a running total even when the individual terms are large. For $u(1),\dots,u(4) = (1,-1,1,-1)$, the running totals are $1,0,1,0$. Against arbitrary readings $x_1,\dots,x_4$,

$$\sum_{n=1}^{4} u(n)x_n = (x_1 - x_2) + (x_3 - x_4).$$

Constant partners cancel completely. Slowly changing partners leave only their changes. Summation by parts extends this calculation to any sequence, with its final running total supplying an endpoint term.

On a uniform periodic grid, the Bernoulli kernels in Theorem 4.1 move the calculation onto point readings, jumps and regular derivatives. For an arbitrary weighted sequence, ordinary summation by parts is the corresponding operation. If $T(n) = \sum_{k=a}^{n} u(k)$, $T(a-1) = 0$, and x_n is any complex sequence, then finite telescoping gives

$$\sum_{n=a}^{b} u(n)x_n = T(b)x_b + \sum_{n=a}^{b-1} T(n)(x_n - x_{n+1}). \tag{10}$$

Indeed, substitute $u(n) = T(n) - T(n-1)$ and shift the second finite sum; its lower endpoint vanishes because $T(a-1) = 0$. For mixed energy take $x_n = v_n\overline{g(n)}$, where v_n is the observation mass. Its difference is

$$x_n - x_{n+1} = (v_n - v_{n+1})\overline{g(n)} - v_{n+1}\overline{g(n+1) - g(n)}.$$

Both the mass variation and the field variation remain.

With an unanchored primitive F, the lower endpoint is $-F(a-1)x_a$. Subtracting $F(a-1)$ from the primitive redistributes that endpoint across the formula. It cannot remove a contribution to the observed energy. At an infinite upper endpoint, write the finite identity first and prove that its endpoint tends to zero.

7.3. **Uniform estimates and signed work.** If $|T(n)| \le P$, (10) gives

$$\left|\sum_{n=a}^{b} u(n)x_n\right| \le P\left(|x_b| + \sum_{n=a}^{b-1} |x_n - x_{n+1}|\right).$$

The primitive bound measures cancellation in u; the bracket measures the variation of its weighted partner. For a growing family, write both as functions of its scale. A small primitive for each component is useful only after the number, amplitudes and interactions of those components have also been counted.

If this absolute estimate is too large, return to the signed identity. Its parent-shell term can be negative. Its mixed work can cancel a new square. Taking absolute values before those terms have been joined changes the estimate being sought.

Problem 14 (Moving an endpoint)**.** Observe a fixed real field on $a < n \le b$ with masses $v_n > 0$. Move the window to $a+1 < n \le b+1$ and add a field g. Write the change without losing either endpoint.

Problem 15 (Weight variation with a constant partner)**.** On $1 \le n \le 5$, take $u(n) = (-1)^{n+1}$, $v(n) = 1$ and $w_n = 1/[n(n+1)]$. Compute the weighted sum directly. Recompute it by summation by parts, using the primitive $T(n) = \sum_{k=1}^{n} u(k)$. Retain the upper endpoint and separate variation of w_n from variation of $v(n)$. Identify the errors in arguments that discard either the endpoint or all interior terms because the partner v is constant.

8. Interaction, refinement and dilation

The opening calculation showed that adding two fields also adds their interaction. We now make that interaction an operation, then track what happens when the sampling grid changes.

For admissible fields f, g on a common boundary partition, define

$$K_{q,\theta}(f,g) = \int_0^1 f\overline{g}\,dx - Q_{q,\theta}(f\overline{g}).$$

For real fields this is simply the discrepancy of their pointwise product. For complex fields the conjugate gives $K(g,f) = \overline{K(f,g)}$, with the grid subscripts understood. The form is linear in its first input; scaling its second input by c scales the result by $\overline{c}$. These properties are called Hermitian symmetry and sesquilinearity. The mixed form need not be positive. Linearity gives

$$D_{q,\theta}[f+g] = D_{q,\theta}[f] + D_{q,\theta}[g] + 2\,\mathrm{Re}\,K_{q,\theta}(f,g). \tag{11}$$

Theorem 4.1 applies directly to $f\overline{g}$, so it also resolves each mixed contribution into boundary and regular terms.

8.1. **Collision matrix.** With many components, repeating a two-field expansion becomes cumbersome. A matrix stores the result of every pairwise comparison. Its diagonal records individual defects, and its off-diagonal entries record the mixed defects. Choosing the component amplitudes then selects one quadratic expression from this table. For two identical real fields of defect d, all four entries are d; amplitudes $(1,1)$ give $4d$, while $(1,-1)$ give zero.

Let $f_1, \dots, f_s$ be admissible. For $z \in \mathbb{C}^s$, put $F_z = \sum_{j=1}^{s} z_j f_j$ and define

$$G^{\mathrm{cont}}_{jk} = \int_0^1 \overline{f_j} f_k\,dx, \qquad G^{\mathrm{grid}}_{jk} = Q_{q,\theta}(\overline{f_j} f_k), \qquad H = G^{\mathrm{cont}} - G^{\mathrm{grid}}.$$

Both Gram matrices are positive semidefinite. Their difference can have either sign.

Proposition 8.1 (Matrix form and change of coordinates)**.** *For every coefficient vector z,*

$$D_{q,\theta}[F_z] = z^* H z.$$

If every f_j takes its midpoint reading at each boundary, the matrix of the sampled point correction is

$$P_{jk} = \frac{1}{4q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} \overline{[f_j]_b} [f_k]_b.$$

In particular, P is positive semidefinite and

$$z^* P z = \frac{1}{4q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} \left| \sum_j z_j [f_j]_b \right|^2 .$$

A unitary change of coefficients $z = Uw$ replaces H and P by U^*HU and U^*PU. It preserves their eigenvalues and traces.

Proof. Expand $|F_z|^2$ before integrating or sampling. This gives z^*Hz. Apply the midpoint identity to F_z at each sampled boundary to obtain z^*Pz. Its displayed expression is a sum of nonnegative squares. Substituting $z = Uw$ gives the transformed matrices. Since $U^*U = I$, they are unitarily similar to the original matrices, which proves the last assertion. □

The matrix formulation separates two useful facts. Collision point corrections are nonnegative for midpoint readings. The full defect also contains the signed matrix $H - P$. A basis change reorganizes both matrices. It does not by itself estimate $H - P$, or require equal energy in different coordinates. Character transforms can be used as unitary coordinate changes when their normalization and underlying finite space are specified; additional symmetry is needed for any stronger conclusion.

8.2. **Grid refinement and dilation.** The six-point grid separates into two three-point grids. Its even indices give $0, 1/3, 2/3$, and its odd indices give $1/6, 1/2, 5/6$. The six-point average is therefore the mean of the two three-point averages. This is the first identity below.

Dilation changes the arguments read by the grid. Sampling $f(2x)$ at those same six addresses reads f at $0, 1/3, 2/3$, each twice. Only three distinct readings remain. In general the greatest common divisor of the dilation and grid size counts these repetitions.

Proposition 8.2 (Grid refinement and dilation)**.** *For a fixed field f, positive integers r, m, q and $0 \le \theta < 1$,*

$$D_{rq,\theta}[f] = \frac{1}{r}\sum_{j=0}^{r-1} D_{q,(j+\theta)/r}[f]. \tag{12}$$

For $\phi \in \mathbb{T}$, define $(U_{m,\phi}f)(x) = f(mx+\phi)$. If $d = (m,q)$, $q' = q/d$ and $m' = m/d$, then

$$D_{q,\theta}[U_{m,\phi}f] = D_{q',\{m'\theta+q'\phi\}}[f]. \tag{13}$$

These identities also hold for any measurable f with finite squared integral and specified finite point readings.

Proof. The rq samples split into the r classes with indices $ra + j$. Their averages are the q-point averages in (12). For dilation, $a \mapsto m'a$ permutes the residues modulo q', each occurring d times as a ranges modulo q. The shifted arguments are therefore the q'-point grid with offset $\{m'\theta + q'\phi\}$. The continuous integral of $|f(mx+\phi)|^2$ equals that of $|f|^2$ by periodicity and substitution. □

For a fixed field, averaging over all grid offsets also gives

$$\int_0^1 D_{q,\theta}[f]\, d\theta = 0.$$

Indeed the substitution $x = (a+\theta)/q$ turns the offset average of $Q_{q,\theta}|f|^2$ into its continuous integral. A persistent defect along a specified aligned grid is compatible with this identity.

As a direct check on the role of resonance, take $F(x) = 2c\psi(mx+\phi)$. Put $\alpha = \{m'\theta + q'\phi\}$ as above. For the unit-frequency field $2c\psi$, the only boundary has $U_0 = 0$, $C_0 = c$ and $V = 2c$. Thus

$$D_{q,\theta}[F] = |c|^2\left(\frac{d}{q}\mathbf{1}_{\{\alpha=0\}} - \frac{4d^2}{q^2}\widetilde{B}_2(\alpha)\right). \tag{14}$$

For $q = m$ and zero phases this is $|c|^2/3$, since every sample is a reset. For fixed m, c, ϕ it tends to zero as $q \to \infty$.

For a unit clock with $c = 1$, the unshifted and half-cell formulas are particularly simple:

$$D_q[2\psi] = \frac{1}{q} - \frac{2}{3q^2}, \qquad D_{q,1/2}[2\psi] = \frac{1}{3q^2}.$$

At offset $\theta = 1/8$, the same field instead has defect $-11/(48q^2)$. These expressions all tend to zero, while having different signs and different leading

orders. The jump at an exactly aligned offset is the contribution of its specified reset reading.

Problem 16 (Dilation of a shifted grid)**.** Let $f = 2\psi$ and $f_r(x) = f(rx)$. On the six-point grid with $\theta = 1/2$, list the distinct arguments of f and their multiplicities for $r = 2$ and $r = 4$. Compute both energy defects. Repeat for $\theta = 0$. Explain why the common value $\gcd(r, 6) = 2$ does not by itself determine the defect on the shifted grid.

8.3. **Joint field and grid refinement.** Equation (12) holds with the field fixed. Suppose the new field is $f + g$ and the new grid has rq points. Combining refinement with polarization gives

$$D_{rq,\theta}[f+g] = \frac{1}{r}\sum_{j=0}^{r-1} D_{q,(j+\theta)/r}[f] + D_{rq,\theta}[g] \\ + 2\operatorname{Re} K_{rq,\theta}(f,g).$$

The new field contributes its own defect and its interaction with the old field. This is the appropriate identity when refinement adds clocks or changes amplitudes. A grid identity alone would omit those terms.

8.4. **Algebraic rules.** Fix the grid and write D and K with its subscripts suppressed. For scalars c, d and admissible fields,

$$\begin{aligned} D(cf) &= |c|^2 D(f), \\ K(cf, dg) &= c\bar{d}\, K(f,g), \\ D(f+g) &= D(f) + D(g) + 2\operatorname{Re} K(f,g). \end{aligned}$$

These follow by expanding the squared magnitude or mixed product before either observation. Point changes use Proposition 3.1. Grid refinement and dilation use Proposition 8.2; translation is its case $m = 1$, with new offset $\{\theta + q\phi\}$. Write

$$P_{q,\theta}(f) = A_{q,\theta}(|f|^2), \qquad R_{q,\theta}(f) = D_{q,\theta}[f] - P_{q,\theta}(f).$$

Then $D = P + R$ separates the sampled point contribution from the remaining signed work. Each rule keeps the same field, grid and reading conventions unless their change is explicitly included.

8.5. **Coordinate and resonance problems.**

Problem 17 (Two identical fields in a different basis)**.** Take $f_1 = f_2 = 2\psi$, sampled on the unshifted q-point grid. Find the defect matrix and diagonalize it with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 18 (Resonance and one extra sample)**.** For $f_m(x) = 2\psi(mx)$, compare the unshifted grids with $q = m$ and $q = m + 1$ as $m \to \infty$.

8.6. **Invariant readings.** An addition can change every sample while leaving one combination of samples fixed. That fixed reading can force energy to remain. Take three equally weighted addresses and the field $f = (0, 1, 0)$. Permit any affine addition $h(i) = a + bi$. The second difference

$$L(f) = f(1) - 2f(2) + f(3)$$

annihilates every such addition, so $L(f + h) = -2$. Cauchy–Schwarz gives

$$|L(f + h)|^2 \leq (1^2 + (-2)^2 + 1^2) \sum_{i=1}^{3} |f(i) + h(i)|^2.$$

Consequently the observed energy is at least $2/3$. The constant addition $h = -1/3$ attains that value, leaving $(-1/3, 2/3, -1/3)$. The permitted additions can remove the affine part of the readings; the remaining bend has a definite energy cost.

Here the weights are one, so energy is a sum rather than a normalized mean. Changing the weights changes the cost of preserving the same reading. With weights $(1, 2, 1)$, the lower bound becomes one and is attained by the addition $h = -1/2$.

Theorem 8.3 (Energy forced by invariant readings)**.** *Let I be a finite address set, let $w_i > 0$, and write $E_w[f] = \sum_{i \in I} w_i |f(i)|^2$. Let V be a linear space of allowed additions. Suppose*

$$L(f) = \sum_{i \in I} a_i f(i), \qquad L(h) = 0 \quad (h \in V), \qquad c_L = \sum_{i \in I} \frac{|a_i|^2}{w_i} > 0.$$

Then, for every $h \in V$,

$$E_w[f + h] \geq \frac{|L(f)|^2}{c_L}. \tag{15}$$

More generally, if nonzero functionals $L_1, \dots, L_k$ annihilate V and have pairwise disjoint supports $I_1, \dots, I_k$, then

$$E_w[f+h] \geq \sum_{\alpha=1}^{k} \frac{|L_\alpha(f)|^2}{\sum_{i \in I_\alpha} |a_{\alpha i}|^2 / w_i}. \tag{16}$$

Proof. The invariant reading is $L(f+h) = L(f)$. Apply Cauchy–Schwarz to $a_i/\sqrt{w_i}$ and $\sqrt{w_i}(f(i)+h(i))$ to obtain (15). For the second assertion apply the same argument on each support. Disjointness allows their energies to be added, and the energy on any unused address is nonnegative. □

This is weighted Cauchy–Schwarz used as a refinement rule. First specify the permitted additions, then construct a reading that all of them annihilate. Its value is inherited from the old field. The denominator records how much energy is required to carry that value under the actual observation weights. No sign condition is imposed on the individual field components.

8.7. **Joint invariant readings.** Several invariant readings may use the same addresses. Their interaction must then be retained, just as it is for the original fields. The task is to find the least energy compatible with all the retained readings at once. Let the rows of a matrix A be their coefficients, so Af is the list of retained values, and let $W = \operatorname{diag}(w_i)$ record the cost of each address. The condition $\ker A = V$ says that these readings detect every change outside the allowed space. When some such changes remain undetected, the same formula still gives a lower bound, but need not be attained by an allowed addition.

Proposition 8.4 (Joint invariant readings)**.** *Suppose the rows of A are linearly independent and $Ah = 0$ for every $h \in V$. Put $G = AW^{-1}A^*$ and $b = Af$. Then G is positive definite and*

$$E_w[f+h] \geq b^* G^{-1} b \qquad (h \in V). \tag{17}$$

If $\ker A = V$, the right side is the minimum energy over all allowed additions. For disjoint supports G is diagonal, and this bound is (16).

Proof. For nonzero c, row independence gives $A^*c \neq 0$. Hence

$$c^* G c = (A^*c)^* W^{-1} (A^*c) > 0.$$

Define $z = W^{-1}A^*G^{-1}b$. Then $Az = b$. For any $h \in V$, put $r = f + h - z$. Since $Ar = 0$, one has $z^*Wr = (G^{-1}b)^*Ar = 0$. Thus

$$E_w[f+h] = E_w[z] + E_w[r] = b^*G^{-1}b + E_w[r].$$

This proves the bound. If $\ker A = V$, then $z - f \in V$ and equality is attained by that addition. Disjoint supports make all off-diagonal entries of G vanish. $\square$

This is the classical minimum-norm problem under linear constraints [4, Sections 6.2 and 10.1.1], written in the coordinates of the invariant readings. The joint matrix measures their overlap in the reciprocal weights. Adding their separate lower bounds without this matrix can count the same retained energy twice.

Problem 19 (Two overlapping second differences)**.** Give four addresses unit weights, take $f = (0, 1, 0, 0)$, and permit all affine additions. Use

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

to find the minimum energy. Compare it with the sum of the two separate reading bounds.

Problem 20 (Weighted adjustment)**.** Give three addresses weights $(1, 2, 1)$ and field $f = (1, 0, 0)$. The permitted additions are constant fields. Find the minimum energy and an addition attaining it. Then obtain the same value from two independent invariant readings and their joint matrix.

Problem 21 (A reading that misses the residual)**.** Give three addresses unit weights, take $f = (0, 1, 0)$, and again permit constant additions. The reading $L(v) = v_1 - v_3$ is invariant and $L(f) = 0$. Does its lower bound determine the minimum energy? Calculate that minimum and explain what the reading fails to observe.

9. Native weights and moving fields

An equally weighted periodic grid and an activated arithmetic tail observe a field in different ways. Summation by parts remains the classical tool. The task here is to carry the actual observation, reset reading and field change through it. Here *native* means that the weights are prescribed by the arithmetic observation.

The first three native masses are $1/2, 1/6, 1/12$. The later readings continue indefinitely, with decreasing mass. A constant unit field has energy one over all positive addresses, but energy $1/2$ after the first address is removed. That change occurs before the field itself has changed. The weights

are

$$w_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \sum_{n\geq a} w_n = \frac{1}{a}.$$

For an integer $S \geq 0$, define the native energy and mixed observation

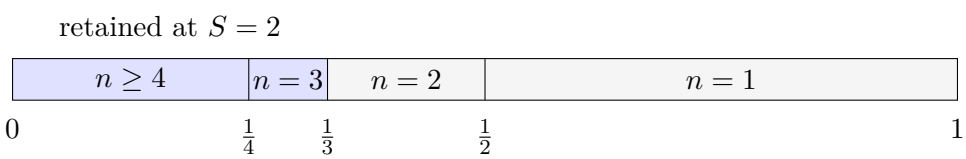
$$E_S[p] = \sum_{n>S} w_n |p(n)|^2, \qquad \langle p, v\rangle_S = \sum_{n>S} w_n p(n)\overline{v(n)}.$$

The cutoff is part of the observation. It excludes S and begins at $S+1$. A finite clock field is bounded, so its energy is finite for every fixed cutoff. No uniform bound in a changing clock family follows from this assertion alone.

The weights also have an exact geometric reading. For the step interpolation $\widehat{p}(u) = p(\lfloor u \rfloor)$,

$$E_S[p] = \int_{S+1}^{\infty} \frac{|\widehat{p}(u)|^2}{u^2}\, du = \int_0^{1/(S+1)} |p(\lfloor 1/x \rfloor)|^2\, dx. \tag{18}$$

Each address cell has mass $\int_n^{n+1} u^{-2}\, du = w_n$. Thus the infinite tail consists of reciprocal cells accumulating at zero. Formula (18) is an identity of observations, not a limiting approximation by equally weighted samples.



The cell $(1/(n+1), 1/n]$ has length w_n. At $S = 2$, the retained cells $n \geq 3$ fill the interval $0 < x \leq 1/3$. The entire infinite tail is present in this finite interval; its cells become progressively shorter near zero. Individual endpoints have no effect on the integral in (18).

9.1. **Approximation of unity.** The reciprocal cells make the unity account of Section 1.3 geometric: they partition $(0, 1]$, and $\mathbf{1}(n) = 1$ is the constant reading on every cell. Thus $E_0[\mathbf{1}] = 1$, whereas $E_S[\mathbf{1}] = 1/(S+1)$. Activation retains the mass of the selected tail. Renormalizing that tail would change the observation.

For $N \geq 2$, let a clock approximation and its residual be

$$p_N(n) = \sum_{j=2}^{N} c_{N,j}\{n/j\}, \qquad r_N(n) = \mathbf{1}(n) - p_N(n), \qquad \mathcal{R}_N = E_0[r_N].$$

The coefficients may change as more clocks become available, while unity remains the same target. Every such finite field is bounded, so its complete error is finite. At a common multiple of its periods, all fractional clocks read zero. The residual there reads one and has positive native mass. Thus every finite clock approximation leaves positive error.

For an added finite clock field h, the new residual is $r_N - h$. Expanding its square gives the complete error change

$$E_0[r_N - h] - E_0[r_N] = E_0[h] - 2\,\mathrm{Re}\langle r_N, h\rangle_0.$$

All terms converge by boundedness and summability of the weights. An improvement requires the residual's signed response to pay for the correction's energy. Several corrections must be combined before their cost is squared. If the old and new coefficients minimize error over nested clock spaces, the minimum cannot increase, because the old fit is still available. A prescribed coefficient change must instead be assessed by the displayed identity.

The first fractional clock. The clock $\phi_2(n) = \{n/2\}$ reads $1/2$ on odd addresses and zero on even addresses, including every reset. The odd addresses have mass $\log 2$. To see the infinite sum directly, for $M \geq 1$ write

$$\sum_{k=1}^{M} w_{2k-1} = \int_0^1 \frac{1 - x^{2M}}{1 + x}\,dx \longrightarrow \int_0^1 \frac{dx}{1 + x} = \log 2.$$

The omitted integral is nonnegative and at most $1/(2M + 1)$. The even addresses therefore have mass $1 - \log 2$. For a real coefficient c, the complete squared error is

$$E_0[\mathbf{1} - c\phi_2] = \log 2\,(1 - c/2)^2 + 1 - \log 2.$$

It is minimized at $c = 2$. The fit equals unity at every odd address; its residual equals unity at every even address. Their energies are $\log 2$ and $1 - \log 2$. This is a partition of the original target over the complete observation, with actual right-hand clock readings at every reset [6].

These comparisons keep the observation E_0 fixed. A moving activation cutoff also removes old readings; Proposition 9.3 retains that additional contribution.

9.2. **Reciprocal-weight boundary terms.** For the constant reading one, the identity $w_n = 1/n - 1/(n + 1)$ makes successive terms cancel. Between a and N only $1/a - 1/(N + 1)$ remains. A changing reading interrupts that cancellation by its successive differences. The next formula retains those interruptions and both endpoints.

Write $\Delta h(n) = h(n) - h(n-1)$ for a backward difference.

Proposition 9.1 (Reciprocal boundary identity)**.** *For arbitrary complex readings $h(a), \dots, h(N)$, with $1 \le a \le N$,*

$$\sum_{n=a}^{N} w_n h(n) = \frac{h(a)}{a} - \frac{h(N)}{N+1} + \sum_{n=a+1}^{N} \frac{\Delta h(n)}{n}. \tag{19}$$

If $\sum_{n \ge a} w_n |h(n)| < \infty$ and $h(N)/(N+1) \to 0$, then

$$\sum_{n=a}^{\infty} w_n h(n) = \frac{h(a)}{a} + \lim_{N\to\infty} \sum_{n=a+1}^{N} \frac{\Delta h(n)}{n}. \tag{20}$$

Bounded h satisfies both hypotheses. The boundary series is taken in increasing address order and need not converge absolutely.

Proof. Expand $w_n = 1/n - 1/(n+1)$ and shift the index in the second finite sum. The lower endpoint is $h(a)/a$, the upper endpoint is $-h(N)/(N+1)$, and the interior terms are the displayed differences. Under the stated hypotheses, the left side converges absolutely and the upper endpoint tends to zero. The finite equality then proves existence and the value of the ordered limit on the right. □

For energy, put $h = |p|^2$ and $a = S+1$. A bounded clock field therefore has a complete boundary account beginning with $|p(S+1)|^2/(S+1)$. The combined differences $\Delta|p|^2$ must be kept together in the ordered limit. Separating a positive jump-square series from its mixed series can produce two divergent sums even when their joined series converges.

For interaction estimates it is more useful to move the difference onto the weighted partner. Define the anchored primitive $T_a(n) = \sum_{k=a}^{n} u(k)$, with $T_a(a-1) = 0$.

Proposition 9.2 (Weighted variation identity)**.** *For $1 \le a \le N$,*

$$\begin{aligned}\sum_{n=a}^{N} w_n u(n) v(n) &= T_a(N) w_N v(N) \\ &\quad + \sum_{n=a}^{N-1} T_a(n)\big((w_n - w_{n+1}) v(n) - w_{n+1} \Delta v(n+1)\big),\end{aligned} \tag{21}$$

where

$$w_n - w_{n+1} = \frac{2}{n(n+1)(n+2)}.$$

Suppose $|T_a(n)| \le B$, $|v(n)| \le V$ *and* $\sum_{n\ge a} w_{n+1}|\Delta v(n+1)| < \infty$. *Then the upper endpoint tends to zero, both infinite sums in* (21) *converge absolutely, and*

$$\left|\sum_{n\ge a} w_n u(n) v(n)\right| \le B\left(V w_a + \sum_{n\ge a} w_{n+1}|\Delta v(n+1)|\right). \tag{22}$$

Proof. Set $x_n = w_n v(n)$ and substitute $u(n) = T_a(n) - T_a(n-1)$ into the finite sum. Index shifting gives $T_a(N)x_N + \sum_{n=a}^{N-1} T_a(n)(x_n - x_{n+1})$. Expansion of $x_n - x_{n+1}$ gives the two terms in (21). If instead $F(n) = \sum_{k=1}^{n} u(k)$ is used, the finite formula includes the lower endpoint $-F(a-1)x_a$:

$$\sum_{n=a}^{N} u(n)x_n = F(N)x_N - F(a-1)x_a + \sum_{n=a}^{N-1} F(n)(x_n - x_{n+1}).$$

Anchoring replaces F by $F - F(a-1)$; it does not discard that endpoint.

For the limit, $|T_a(N)x_N| \le BVw_N \to 0$ and $\sum_{n\ge a}(w_n - w_{n+1}) = w_a$. These facts and the assumed weighted variation prove absolute convergence of the right side and the stated bound. Also $|u(n)| \le 2B$ (and $|u(a)| \le B$), so the original weighted product converges absolutely as well. □

For a conjugate mixed observation, replace v by $\overline{v}$. The first interior term is caused by variation of the weights. It is present even when the partner field is constant.

9.3. **Right-hand clocks and combined resets.** At integer addresses, the clock $\{u/3\}$ repeats the readings $0, 1/3, 2/3$. At three it resets to zero, while its left trace is one. A sum of such clocks is specified by a coefficient table, called its source. Here $q(j)$ is the coefficient of the clock of length j; it is a function of j, distinct from the grid size q used earlier. At address n, precisely the lengths dividing n reset together.

Let $q(j)$, $j \ge 1$, be a finitely supported complex source. Define

$$p(u) = \sum_j q(j)\{u/j\}, \qquad \sigma = \sum_j \frac{q(j)}{j}, \qquad c(n) = \sum_{j|n} q(j).$$

At an integer address n, the actual reading is the right-hand value. The traces and discrete increment satisfy

$$p(n) = p(n+), \qquad p(n-) = p(n) + c(n), \qquad \Delta p(n) = \sigma - c(n). \tag{23}$$

Indeed each fractional clock rises by $1/j$ between successive integer addresses and subtracts one exactly when $j \mid n$. If the source is reciprocally neutral, $\sigma = 0$. The slope then vanishes on every open unit cell, $p(n-) = p(n-1)$, and the combined signed reset is $\gamma(n) = -c(n)$.

For a neutral source the complete energy jump is

$$\Delta|p(n)|^2 = 2\,\mathrm{Re}\big(\overline{p(n-1)}\gamma(n)\big) + |\gamma(n)|^2. \tag{24}$$

The divisor contributions are added in $c(n)$ before taking a square. Its square contains every interaction between simultaneous resets. The other term is their work against the inherited reading. Substituting their joined expression into (20) preserves the ordered boundary account.

The same rule for two complex fields is

$$\begin{aligned}\Delta(X\overline{Y})(n) = X(n-1)\overline{\Delta Y(n)} + \overline{Y(n-1)}\Delta X(n)\\ + \Delta X(n)\overline{\Delta Y(n)}.\end{aligned}$$

It retains both inherited readings and the simultaneous jump product. All these identities follow by expanding the new reading as the old reading plus its increment.

9.4. **Point-value correction at resets.** For two sequences with finite native energy, the point-value replacement rule is

$$E_S[g] - E_S[f] = \sum_{n>S} w_n\big(|g(n)|^2 - |f(n)|^2\big). \tag{25}$$

Only changed readings contribute. The series is absolutely convergent, since its absolute terms are bounded by the two energy series. This extends Proposition 3.1 to the weighted infinite set. The sign is the change in sampled energy; it is opposite to the change in an integral-minus-sample defect when the integral is fixed.

At a reset put $m_n = (p(n+) + p(n-))/2$ and $\gamma_n = p(n+) - p(n-)$. The actual right reading is $m_n + \gamma_n/2$, so

$$|p(n+)|^2 - |m_n|^2 = \mathrm{Re}(m_n\overline{\gamma_n}) + \frac{1}{4}|\gamma_n|^2, \tag{26}$$

$$a_n^+(|p|^2) := \frac{|p(n-)|^2 + |p(n+)|^2}{2} - |p(n+)|^2 = -\,\mathrm{Re}(m_n\overline{\gamma_n}). \tag{27}$$

The midpoint correction $|\gamma_n|^2/4$ has been adjusted by the change in the squared reading. The resulting right-hand correction can have either sign. For a neutral finite clock field, both the midpoint and right-hand sequences are bounded, so (25) applies to all their reset addresses. The midpoint is

formed from the same clock field, not by silently replacing its fractional clocks with centered clocks.

Activation has its own boundary. If $b_S(n) = \mathbf{1}_{\{n>S\}}p(n)$, then for $n \geq 1$

$$\Delta b_S(n) = \mathbf{1}_{\{n=S+1\}}p(S+1) + \mathbf{1}_{\{n\geq S+2\}}\Delta p(n). \tag{28}$$

The first admitted reading is not generally the raw clock's reset at that address. The activation term records the whole admitted field.

9.5. **Field increments and activation.** There are two changes to follow. The moved cutoff removes old readings on $S < n \leq T$. Beyond T, the old field is still present when the addition arrives. Subtracting the removed energy and expanding the retained square gives the complete change. In particular, a positive new square can be outweighed by the removed shell or a negative interaction.

Proposition 9.3 (Complete moving-field work)**.** *Let $0 \leq S \leq T$ be integers, let $E_S[p]$ and $E_T[p+v]$ be finite, and interpret all fields on the common tail $n > T$. Then*

$$\begin{aligned} E_T[p+v] - E_S[p] = &- \sum_{S<n\leq T} w_n|p(n)|^2 \\ &+ \sum_{n>T} w_n \left(2\,\mathrm{Re}(\overline{p(n)}v(n)) + |v(n)|^2\right). \end{aligned} \tag{29}$$

All terms converge absolutely. If $v = d + e$ with $E_T[d], E_T[e] < \infty$, its tail integrand is

$$2\,\mathrm{Re}(\overline{p}d) + 2\,\mathrm{Re}(\overline{p}e) + |d|^2 + 2\,\mathrm{Re}(\overline{d}e) + |e|^2. \tag{30}$$

Proof. Split the old energy into $S < n \leq T$ and $n > T$. On the common tail, expand $|p+v|^2 - |p|^2$. Since $|v|^2 \leq 2|p+v|^2 + 2|p|^2$, the change v has finite tail energy. Cauchy–Schwarz makes the mixed product absolutely summable. The second expansion follows by substituting $v = d + e$. The same Cauchy–Schwarz argument proves convergence of each separate term. □

The negative parent shell contains the square of the entire old field. If that field is a sum of families, their shell interactions are inside the same square. If the change is a sum, its square likewise contains all interactions between changed families. A shorter observation window or a separate bound for the new square does not replace (29).

9.6. **Two-clock native tail.** Consider

$$p(u) = \{u\} - 2\{u/2\}.$$

Its reciprocal source sum is $1 - 2/2 = 0$. At integer addresses, $p(n) = -1$ for odd n and $p(n) = 0$ for even n. The formal clock of length one has zero reading at every integer, but its source coefficient is needed for neutrality and for the reset formula.

At address one the left and right readings are $0, -1$. The midpoint reading is $-1/2$, with midpoint energy correction $1/4$. The actual right-hand correction is $1/2 - 1 = -1/2$. At address two the readings are $-1, 0$ and the right-hand correction is $1/2$. Thus positive midpoint corrections cannot be used unchanged even in this two-clock example.

The complete energy is

$$E_0[p] = \sum_{r \geq 0} \frac{1}{(2r+1)(2r+2)} = \log 2.$$

To evaluate the tail exactly, write each summand as $\int_0^1 x^{2r}(1-x)\,dx$. Nonnegativity permits summation under the integral, giving $\int_0^1 (1+x)^{-1}\,dx = \log 2$. Now double the field and move the cutoff from $S = 0$ to $T = 1$. The change is $v = p$. Formula (29) gives

$$E_1[2p] - E_0[p] = -\frac{1}{2} + 2\left(\log 2 - \frac{1}{2}\right) + \left(\log 2 - \frac{1}{2}\right) = 3\log 2 - 2.$$

The three terms are the removed parent shell, the inherited interaction, and the new square. Directly, $E_1[2p] = 4(\log 2 - 1/2)$, which gives the same answer. Every later address is included in both calculations. Here every raw jump has squared magnitude one, so the separate boundary series $\sum_n |\gamma(n)|^2/n$ diverges. Its mixed counterpart cancels that divergence in the ordered energy-jump series. The joined boundary expression is the convergent object.

9.7. **Weighted tail localization.** The same two-clock example illustrates how a finite calculation can enclose a complete observation. Stop its changed energy at $N = 4$. The removed parent reading contributes $-1/2$, and the only admitted odd address is three. The finite answer is

$$H_4 = -\frac{1}{2} + \frac{3}{3 \cdot 4} = -\frac{1}{4}.$$

Every omitted term lies between zero and $3w_n$. The remaining weight is $1/(N+1)$, giving

$$-\frac{1}{4} \le 3\log 2 - 2 \le -\frac{1}{4} + \frac{3}{5} = \frac{7}{20}.$$

Stopping at eight gives $H_8 = -27/280$ and the interval $[-27/280, 199/840]$. The finite answer may have the wrong sign while its enclosure remains valid. A tail allowance is part of the answer.

This calculation extends to an arbitrary positive observation on the integer addresses. The old and new fields can grow with an outside parameter; their envelopes determine how far the observation must run.

Theorem 9.4 (Weighted localization)**.** *Let $w_n \ge 0$ be finite, and let p, q be complex fields with finite readings. Fix integers $0 \le S \le T \le N$. Suppose that, for $n > N$,*

$$|p(n)| \le A_n, \qquad |q(n)| \le C_n, \qquad A_n, C_n \ge 0,$$
$$L_N = \sum_{n>N} w_n A_n^2 < \infty, \qquad U_N = \sum_{n>N} w_n C_n^2 < \infty.$$

Write $v = q - p$ and

$$H_N = -\sum_{S<n\le T} w_n |p(n)|^2 + \sum_{T<n\le N} w_n \big(2\,\mathrm{Re}(\overline{p(n)}v(n)) + |v(n)|^2\big).$$

Both complete energies are finite, and their change satisfies

$$H_N - L_N \le \sum_{n>T} w_n |q(n)|^2 - \sum_{n>S} w_n |p(n)|^2 \le H_N + U_N. \tag{31}$$

For native weights and constant envelopes $A_n = A$, $C_n = C$, the omitted change lies in $[-A^2/(N+1), C^2/(N+1)]$.

Proof. The finite portions have finitely many finite readings. The two tail energies converge by the stated envelope bounds. Subtracting their absolutely convergent sums gives

$$\sum_{n>T} w_n |q(n)|^2 - \sum_{n>S} w_n |p(n)|^2 = H_N + \sum_{n>N} w_n \big(|q(n)|^2 - |p(n)|^2\big).$$

At each address the last integrand lies between $-|p(n)|^2$ and $|q(n)|^2$. Summing these inequalities proves (31). For native weights, $w_n = 1/n - 1/(n+1)$ telescopes to the stated remaining mass. □

The square of the complete new field gives the upper allowance; the square of the complete old field gives the lower allowance. Expanding the omitted work and bounding its mixed terms separately would usually give a larger enclosure. In the two-clock calculation, the known sign and the identity $|2p|^2 - |p|^2 = 3|p|^2$ improve both sides further.

For a family with envelopes A_S, C_S, an error at most ϵ is guaranteed by

$$N \geq T, \qquad N + 1 \geq \frac{\max(A_S^2, C_S^2)}{\epsilon}.$$

An envelope proportional to S therefore permits localization at a constant multiple of S^2. The cutoff grows with the field. No fixed-field quadrature constant is being used as a uniform estimate.

Problem 22 (A constant field with a moved cutoff)**.** Use native weights, old field $p(n) = 1$, change $v(n) = -1/2$, and cutoffs $S = 1$, $T = 3$. Calculate the complete energy change. Resolve it into the removed shell, inherited interaction and new square. All infinite tails must be evaluated.

Problem 23 (Choosing a sufficient truncation)**.** Suppose $|p(n)| \leq 2$ and $|q(n)| \leq 3$ on the omitted native tail, and the new activation cutoff is four. Find the least integer N for which the envelope bound guarantees absolute tail error at most $1/10$. If the finite head at that cutoff is $-1/10$, which sign conclusion is justified? Repeat the cutoff calculation for envelopes $2S, 3S$ and activation $2S$, where $S \geq 1$ is an integer.

10. Growing signals in weighted windows

Consider a receiver whose observation window moves farther along a sequence while a rapidly alternating component becomes larger. Each sample has a specified mass. The question is whether the observed energy has a stable limit, and whether the oscillation's interaction with the background survives.

The observation in this model has its own weights, specified below; they are not replacements for the native masses. Fix positive integers $a < b$, a bounded weight $\rho \geq 0$, and real functions A, B on $[a, b]$. At scale N observe the addresses $aN < n \leq bN$ with masses

$$v_{N,n} = N^{-2}\rho(n/N).$$

The total mass is $O(N^{-1})$. Let $u(n)$ be a real carrier signal and set the actual readings

$$f_N(n) = B(n/N) + \sqrt{N}\, A(n/N)u(n).$$

The carrier term has growing amplitude. Its square is multiplied by an observation whose mass shrinks at the compensating rate.

10.1. **Four-sample window calculation.** Take $a = 1$, $b = 2$, $\rho(x) = x^{-2}$ and $A = B = 1$, with $u(n) = (-1)^n$. The masses are $v_{N,n} = 1/n^2$. At $N = 4$ the four addresses are $5, 6, 7, 8$, their readings are $-1, 3, -1, 3$, and

$$E_4 = \frac{1}{25} + \frac{9}{36} + \frac{1}{49} + \frac{9}{64}.$$

The result is $35361/78400$. A continuous-address version holds each reading on $[n, n+1)$, so these are its actual right-hand readings. Write $U_N = \sum_{N<n\le 2N} n^{-2}$ and $V_N = \sum_{N<n\le 2N}(-1)^n n^{-2}$. Then exactly

$$E_N = (N+1)U_N + 2\sqrt{N}\,V_N. \tag{32}$$

The first part contains the background square and the carrier square. The second retains their signed interaction.

At $N = 9$ the window becomes $10, \dots, 18$ and the readings alternate between 4 and -2. To compare these observations, extend the formula $f_N(n) = 1+\sqrt{N}(-1)^n$ to all positive addresses. The change field is $f_9 - f_4 = (-1)^n$. Formula (3) gives

$$E_9 - E_4 = -E_4 + \sum_{n=10}^{18} \frac{f_4(n)^2}{n^2} + 2\sum_{n=10}^{18} \frac{f_4(n)(-1)^n}{n^2} + \sum_{n=10}^{18} \frac{1}{n^2}.$$

The removed old window, admitted old readings, mixed term and new square all appear. The windows happen to be disjoint here; the general identity also handles overlap and changes of weight on retained addresses.

The estimate below is an application of classical summation by parts and bounded-variation quadrature. Its role is to expose every constant for this moving observation, so the growing amplitude can be handled without a fixed-field assumption.

10.2. **Uniform bound for growing amplitude.** The slowly varying envelopes also need a quantitative description. Total variation adds the magnitudes of successive changes along a partition and takes the supremum over all partitions. A monotone function has variation equal to its endpoint difference in magnitude; for example, x^{-2} on $[1, 2]$ has variation $3/4$. Oscillation and jumps both consume variation. This is the quantity that the primitive estimate charges to the partner of the rapidly alternating carrier.

For a specified function h of bounded variation on $[a,b]$, put

$$\|h\|_{\mathrm{BV},*} = |h(b)| + \mathrm{Var}_{[a,b]}\, h.$$

Variation includes the assigned point readings. This convention allows step envelopes and exceptional endpoint values. Suppose the three products

$$h_0 = \rho B^2, \qquad h_1 = \rho AB, \qquad h_2 = \rho A^2$$

are of bounded variation. Their Lebesgue integrals are well defined. They correspond to the background square, the mixed product and the carrier square. The constants P and R below control the running sums of the carrier and of its squared fluctuation about c. For the alternating carrier, the first running sum stays small and the squared fluctuation vanishes identically. These two controls serve different terms in the energy expansion.

Theorem 10.1 (Growing carrier under nonuniform observation)**.** *Suppose there are constants $P, R < \infty$ and $c \geq 0$ such that for every finite integer interval I,*

$$\left|\sum_{n\in I} u(n)\right| \leq P, \qquad \left|\sum_{n\in I} (u(n)^2 - c)\right| \leq R.$$

Let $E_N = \sum_{aN<n\leq bN} v_{N,n} f_N(n)^2$. Then

$$\left|E_N - c\int_a^b h_2(x)\,dx - \frac{1}{N}\int_a^b h_0(x)\,dx\right| \leq \frac{c\,\mathrm{Var}(h_2) + R\|h_2\|_{\mathrm{BV},*}}{N} + \frac{2P\|h_1\|_{\mathrm{BV},*}}{N^{3/2}} + \frac{\mathrm{Var}(h_0)}{N^2}. \tag{33}$$

In particular the energies converge to $c\int_a^b \rho A^2$. The constants are independent of N.

Proof. Expanding the square gives

$$E_N = \frac{1}{N^2}\sum h_0(n/N) + \frac{2}{N^{3/2}}\sum u(n)h_1(n/N) + \frac{1}{N}\sum u(n)^2 h_2(n/N),$$

with every sum over $aN < n \leq bN$. On a cell $[(n-1)/N, n/N]$, the difference between the right-hand reading $h(n/N)$ and $h(x)$ is bounded by the variation of h on that cell. Integration and addition give

$$\left|\frac{1}{N}\sum_{aN<n\leq bN} h(n/N) - \int_a^b h(x)\,dx\right| \leq \frac{\mathrm{Var}(h)}{N}.$$

This also holds with specified exceptional point readings, since their changes contribute to the variation.

Anchor the primitive of u at aN. Formula (10) and the first interval hypothesis yield $|\sum u(n)h_1(n/N)| \le P\|h_1\|_{\mathrm{BV},*}$. Apply the same argument to u^2-c to obtain $|\sum(u(n)^2-c)h_2(n/N)| \le R\|h_2\|_{\mathrm{BV},*}$. The two quadrature bounds for h_0, h_2, together with these two primitive bounds, give (33). □

The alternating carrier has $P = 1$, $c = 1$, $R = 0$. For the four-sample example all three products equal x^{-2} on $[1, 2]$. Their variation is $3/4$, their $\|\cdot\|_{\mathrm{BV},*}$ is one, and their integral is $1/2$. Thus every positive integer N satisfies

$$\left|E_N - \frac{1}{2} - \frac{1}{2N}\right| \le \frac{3}{4N} + \frac{2}{N^{3/2}} + \frac{3}{4N^2}. \tag{34}$$

The field grows as $\sqrt{N}$ while its observed energy tends to $1/2$. Its mixed interaction tends to zero at the explicitly controlled rate $N^{-3/2}$. This is a bound for the growing family itself.

Every bounded periodic carrier with zero mean satisfies the first primitive hypothesis. If c is its period mean square, then u^2-c also has zero mean and a bounded primitive. Discontinuous envelopes are allowed as long as the three displayed products have bounded variation. The observation, carrier cancellation and envelope variation have separate roles in the estimate.

Problem 24 (The amplitude threshold)**.** Replace $\sqrt{N}$ by N^α in the alternating example. For which α does the energy vanish, remain finite and nonzero, or grow without bound?

Problem 25 (A carrier with a three-step period)**.** Let $u(3k+1) = 1$, $u(3k+2) = -1$, $u(3k+3) = 0$. In the moving-window model take $a = 1$, $b = 2$ and $\rho = A = B = 1$. Find admissible constants P, R, c from one period and derive an explicit estimate for E_N. State the limit and calculate E_4 directly.

11. Limit regimes

The opening ramp already gave two different limits. Refining its grid with the field fixed sent the defect to zero. Multiplying its amplitude by $\sqrt{q}$ made the defect tend to one. The notation $O_f(q^{-1})$ below means a bound C_f/q with a constant allowed to depend on the entire field. To insert a changing field f_q, we must first determine how C_{f_q} grows. The subscript records a substantive dependence.

Corollary 11.1 (A fixed field)**.** *For every fixed admissible field f,*

$$\sup_{0\le\theta<1} |D_{q,\theta}[f]| = O_f(q^{-1}).$$

If $h = |f|^2$ has no value jumps and its assigned values agree with its continuous traces, the bound improves to $O_f(q^{-2})$.

Proof. There are finitely many boundaries. Their point corrections, value jumps and derivative jumps are fixed, and the regular curvature has finite integral. Bound the point sum by $q^{-1}\sum_b |a_b(h)|$ and use Corollary 4.3 with $c = 0$. Under the stronger hypotheses the point and value-jump terms vanish, leaving only terms of order q^{-2}. □

Thus a nonzero persistent defect requires a changing family within these hypotheses. The relevant growth may occur in collision amplitude, the number of boundaries, or the remaining energy data. For example, a sampled collision of m_q equal unit clocks contributes m_q^2/q. Its natural constant scale is m_q comparable to $\sqrt{q}$. The carry family has $m_q = p - 1$ and $q = p^2$ at the origin. The pulse family obtains a constant scale through amplitudes comparable to $\sqrt{q}$ at a fixed number of endpoints.

For a growing periodic family, write the energy decomposition as

$$D_{q,\theta}[f] = P_{q,\theta}(f) + R_{q,\theta}(f),$$

where R consists of the last three terms of (4). A limit for the point correction becomes a limit for the whole defect precisely when the remaining signed contribution vanishes. Corollary 4.3 gives one sufficient route; the carry family requires a more detailed arithmetic estimate.

Refinement does not imply monotonicity of the defect. A decomposition into components must retain the mixed form K. Either description requires the complete boundary readings and a bound on the signed remainder for the growing family.

11.1. **Absolute bounds and signed cancellation.** For a general field, the absolute estimate in Corollary 4.3 is the first test. If its right side tends to zero, the point correction determines the limit. If it does not, retain the signed boundary sums before taking absolute values. The failure of that sufficient test does not imply that the residual stays large.

For clocks, constant regular curvature removes the integral term completely. The remaining work is then finite and explicit: estimate the correlation between the background and the reset, weighted by ψ, and the

correlation between the slope and the reset, weighted by $\widetilde{B}_2$. An exact evaluation of the latter often reduces the question to the former, as the carry applications below demonstrate.

For a sum, the complete field and its interaction matrix retain the same cross terms. For a transformed grid, apply the gcd and offset rule before counting sampled boundaries.

11.2. **Growing pulses with moving overlap.** Two channels can each have zero sampling defect while their difference has a persistent defect. The following complete calculation joins the point conventions, growing amplitude and mixed energy developed above. It uses only interval counting.

Let q be an even positive integer, put $\delta_q = 1/(2q)$, and define half-open gates on the periodic interval by

$$I = [0, 1/2), \qquad J_q = [\delta_q, 1/2 + \delta_q), \qquad A_q = \sqrt{q}\,\mathbf{1}_I, \quad B_q = \sqrt{q}\,\mathbf{1}_{J_q}.$$

The observed signal is $F_q = A_q - B_q$. One gate is delayed by half a grid cell. Determine the energy seen by the integral and by $X_{q,\theta}$, including the offset $\theta = 1/2$ itself.

Combined field. The two channels cancel on their overlap. Consequently

$$F_q = \sqrt{q}\,\mathbf{1}_{[0,\delta_q)} - \sqrt{q}\,\mathbf{1}_{[1/2,1/2+\delta_q)}.$$

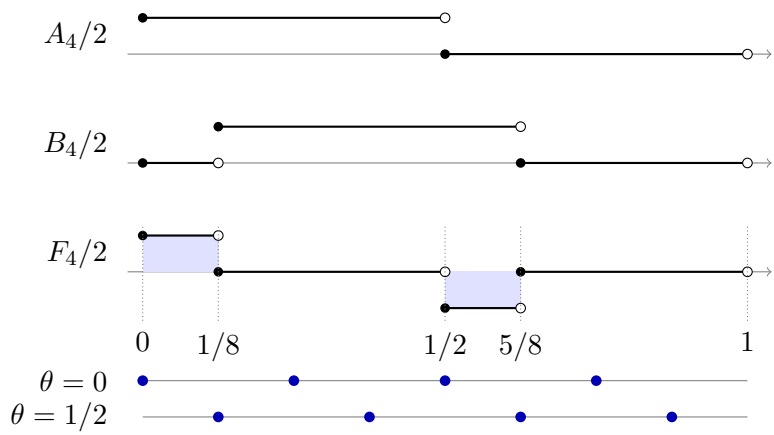
Its square is q on two intervals of length $1/(2q)$ and zero elsewhere. Thus its continuous energy is one for every even q.

For $0 \le \theta < 1/2$, precisely the samples with indices 0 and $q/2$ lie in these two intervals. The sampled energy is two. For $1/2 \le \theta < 1$, both intervals are missed and the sampled energy is zero. At equality the samples lie at the excluded right endpoints. The assigned half-open readings therefore give

$$D_{q,\theta}[F_q] = \begin{cases} -1, & 0 \le \theta < 1/2, \\ 1, & 1/2 \le \theta < 1. \end{cases} \tag{35}$$

There is no asymptotic remainder in this answer. In particular the two constant limits persist as q grows through the even integers.

At $q = 4$, the delay is $1/8$ and the nonzero amplitudes are $2, -2$. The picture shows how subtracting the overlapping gates leaves two short pieces, then places both grids beneath them.



The curves are divided by two to show their shapes. Filled endpoints are included and open endpoints are excluded. The bottom two rows show sample locations. The unshifted grid meets the two nonzero pieces at their included left endpoints; the half-cell grid meets their excluded right endpoints.

The full calculation can be inspected in four readings.

Offset	Readings of F_4	Sampled energy	Defect
0	$(2, 0, -2, 0)$	2	-1
1/2	$(0, 0, 0, 0)$	0	1

Mixed-energy calculation. Each gate has length 1/2 and contains exactly $q/2$ grid samples, for every offset. Hence each channel has continuous and sampled energy $q/2$, and $D_{q,\theta}[A_q] = D_{q,\theta}[B_q] = 0$. The continuous mixed energy is

$$\int_0^1 A_q B_q \, dx = q(1/2 - \delta_q) = q/2 - 1/2.$$

Their sampled mixed energy is the number of grid points in $[\delta_q, 1/2)$. That number is $q/2 - 1$ when $\theta < 1/2$ and $q/2$ when $\theta \geq 1/2$. Therefore

$$K_{q,\theta}(A_q, B_q) = \begin{cases} 1/2, & 0 \leq \theta < 1/2, \\ -1/2, & 1/2 \leq \theta < 1, \end{cases} \qquad D_{q,\theta}[F_q] = -2K_{q,\theta}(A_q, B_q).$$

This recovers (35). Adding the two separate channel defects would give zero and lose the entire answer.

The combined-field and mixed-energy calculations give the same answer. The first locates the two short intervals; the second attributes the full defect

to the interaction. The endpoint readings determine which grid offsets see that energy.

11.3. **Method selection and extension.** The problems below ask for a choice of method and a transfer of the completed example. A solution should name the object that the chosen operation acts on before estimating it.

Problem 26 (Choose the first operation)**.** For each task, identify the first identity or bound to use and the data it requires. An isolated sample reading changes; an old periodic carrier is paired with a slowly varying weighted partner; a finite energy change has been calculated but its infinite tail remains; a fixed-field error bound is proposed for a sequence whose amplitude grows. In the last case, state what must be checked before the bound can be used uniformly.

Problem 27 (Change the two channel amplitudes)**.** In Section 11.2, replace F_q by $aA_q + bB_q$, where a, b are fixed real numbers. Find its continuous energy, sampled energy and defect for both offset ranges. Determine which choices make the defect zero for every offset. Recover the difference example and also calculate the sum of the channels.

Problem 28 (Growing amplitude on an infinite tail)**.** Fix $a \in \mathbb{R}$ and, for integers $N \geq 1$, put $p_N(n) = a + \sqrt{N}(-1)^n$. Observe $n > N$ with the native weights and write $\mathcal{E}_N = E_N[p_N]$. Separate the two squares and mixed term, bound the complete alternating tail, and find the limit of $\mathcal{E}_N$ with an explicit error allowance. For $a = 1$, choose an integer cutoff $M_N \geq N$ that guarantees the omitted energy is at most $1/N$ using only the uniform envelope for p_N. Explain how the cutoff must respond to the growing amplitude.

Part 3

Finite arithmetic tools

Arithmetic data offer two useful ways to organize the preceding energy calculations. When components are specified by congruences, construct their observed intersections and use them as entries of a matrix. Sections 12 and 13 develop this route, including the effect of moving a boundary or changing the modulus of a phase.

When a kernel repeats under common dilation, group its repeated pairs into primitive shells. Sections 14 and 15 develop this second route. Logarithmic weighting connects the resulting prefixes to prime counting; recovering estimates from that identity requires control of its signed inverse. The two routes make arithmetic structure available to the field and observation calculations already established.

12. Signed collision registers

A congruence records where two arithmetic components can meet. An observation records which of those meetings are actually read. Keeping these two pieces of data together turns a collision count into a finite field to which the energy rules already apply.

Consider the two conditions

$$r \equiv 4 \pmod 7, \qquad r \equiv 1 \pmod 8.$$

They have the common residue 25 modulo 56. An observation on $3 < r < 6$ sees no common reading; one on $24 < r < 27$ sees exactly one. Compatibility of the congruences is therefore only the first part of an interaction calculation. The interval and its weights determine whether that interaction contributes.

12.1. **Incidences and the observation kernel.** Let $\mathcal{A}$ be a finite set of arithmetic components. A component ξ has modulus $c_\xi \geq 1$, residue ρ_ξ, real amplitude z_ξ and orientation $\sigma_\xi \in \{1, -1\}$. Define its incidence and the combined row by

$$I_\xi(r) = \mathbf{1}_{r \equiv \rho_\xi \pmod{c_\xi}}, \qquad \Delta(r) = \sum_{\xi \in \mathcal{A}} \sigma_\xi z_\xi I_\xi(r).$$

Give the integer rows nonnegative weights $v(r)$ of finite support. The energy and intersection kernel are

$$M_v = \sum_r v(r)\Delta(r)^2, \qquad K_v(\xi, \eta) = \sum_r v(r) I_\xi(r) I_\eta(r). \tag{36}$$

Think of each component as a column and each address r as a row. The entry $I_\xi(r)$ records whether that component is present there; its signed amplitude determines its contribution to the reading. Several columns can contribute to the same observed row. In this section $\Delta(r)$ names that combined reading; it is not the backward difference $\Delta h(n)$ used in the weighted identities.

Proposition 12.1 (Signed incidence formula)**.** *The complete energy is*

$$M_v = \sum_{\xi,\eta\in\mathcal{A}} \sigma_\xi \sigma_\eta z_\xi z_\eta K_v(\xi,\eta). \tag{37}$$

The sum is over ordered pairs. In particular, both mixed orientations are retained.

Proof. Expand the square defining M_v and interchange the finite sums. The coefficient of $\sigma_\xi\sigma_\eta z_\xi z_\eta$ is exactly the second expression in (36). □

This is the collision-matrix calculation of Section 8, with its entries evaluated by congruences. The matrix K_v is positive semidefinite because it is a Gram matrix of incidence vectors. Its signed quadratic form nevertheless contains negative mixed terms. Replacing those terms by absolute values changes the energy being estimated.

12.2. **Common residues and finite intervals.** The Chinese remainder theorem supplies the common residue [1, Chapter 5]. Parentheses denote the greatest common divisor and brackets the least common multiple. Write

$$g = (c_1, c_2), \qquad \ell = [c_1, c_2].$$

The notation $g \mid d$ means that g divides d. A bar denotes an inverse in the displayed modulus. The two incidences meet exactly when $g \mid \rho_2 - \rho_1$. If they meet, solve

$$t \equiv \frac{\rho_2-\rho_1}{g}\,\overline{(c_1/g)} \pmod{c_2/g}, \qquad R \equiv \rho_1 + c_1 t \pmod{\ell}, \quad 0 \le R < \ell.$$

For $c_2/g = 1$ take $t = 0$. These formulas follow by substituting $r = \rho_1 + c_1 t$ into the second congruence. The reduced moduli are coprime, so the displayed inverse exists. Every common row is $R + j\ell$.

If the observation is supported on $Y < r < 2Y$ and $\ell > Y$, there is at most one common row. Its first possible representative is

$$j_0 = \left\lfloor \frac{Y-R}{\ell} \right\rfloor + 1, \qquad r_0 = R + j_0\ell.$$

The kernel is $v(r_0)$ when compatibility holds and $r_0 < 2Y$; otherwise it is zero. The inequality $r_0 > Y$ is already enforced. For a longer interval, retain the full sum $\sum_{j:\, Y<R+j\ell<2Y} v(R+j\ell)$.

Problem 29 (A shared row with opposing amplitudes)**.** On $24 < r < 27$ give each integer row weight one. Use the residues 4 modulo 7 and 1 modulo 8, amplitudes $2, 3$, and orientations $+1, -1$. Compute the energy and its four ordered terms.

12.3. **A divisibility register.** A useful arithmetic instance fixes an odd prime p and uses moduli $p/2 < c < p$ and columns $1 \le b < p$. Put

$$\Delta[f](r) = \sum_c a_c \sum_{1\le b<p} f_b\big(\mathbf{1}_{c|p^2-br} - \mathbf{1}_{c|p^2+br}\big), \qquad 1 \le r < p, \tag{38}$$

where a_c, f_b are real coefficients. The row weights $v(r)$ still specify the observation; a_c and f_b specify the field. A column b is a unit modulo c when $(b, c) = 1$. Then its inverse $\bar{b}$ satisfies $b\bar{b} \equiv 1 \pmod{c}$. A unit component (c, b, σ) has residue $\rho \equiv \sigma p^2 \bar{b} \pmod{c}$. A nonunit b has no incidence: any common divisor of b, c would divide p^2, although $c < p$. For two unit components, compatibility becomes

$$(c_1, c_2) \mid \sigma_1 b_2 - \sigma_2 b_1. \tag{39}$$

Multiplying the common-residue condition by the unit $b_1 b_2 \overline{p^2}$ proves this equivalence. The interval test is still needed after (39).

Let $d_c(r)$ denote the c term in (38). If $|a_c| \le A$, $|f_b| \le B$ and $\sum_r v(r) \le V$, then

$$0 \le \sum_r v(r) \sum_c d_c(r)^2 \le 16A^2B^2pV. \tag{40}$$

For fixed c, r each sign permits at most two columns, since $c > p/2$. Hence $|d_c(r)| \le 4AB$, and summing over fewer than p moduli proves the bound. This retains all pairs at a common modulus, including distinct columns.

The remaining energy is the signed sum over different moduli. A first-moment target may require less than control of that energy: Cauchy–Schwarz gives

$$\left|\sum_r W(r)\Delta(r)\right|^2 \le \left(\sum_r |W(r)|\right)\left(\sum_r |W(r)|\Delta(r)^2\right).$$

The implication goes from a second-moment bound to a first-moment bound. A row pair $T, -T$ has zero unweighted sum and energy $2T^2$, so the converse need not hold.

Problem 30 (Compatibility after a sign change)**.** Keep the first residue 4 modulo 7 and replace the second residue 1 modulo 8 by -1 modulo 8. Find the common residue modulo 56. Compare the two intersection kernels on $24 < r < 27$.

Problem 31 (Weighted intersections over several periods)**.** Use components $r \equiv 1 \pmod 3$ and $r \equiv 2 \pmod 4$, with signed amplitudes $2, -1$. Observe the rows $1 \le r \le 25$, giving row ten weight two and every other row weight one. Find the full 2×2 incidence Gram matrix and the observed energy. Recompute the energy after changing the second amplitude to $+1$. Explain why an at-most-one-common-row shortcut would fail for this observation.

13. Boundary localization and reciprocal transfer

Moving the edge of a register changes only the columns crossed by that edge. The cost can be counted before any smooth approximation is chosen. For example, at $p = 11, r = 4$, changing column $b = 10$ by h changes (38) by

$$h(a_9 - a_7).$$

Indeed $p^2 - br = 81$ and $p^2 + br = 161$ have, respectively, the divisors 9 and 7 in $11/2 < c < 11$. The sign and the two modulus coefficients remain visible in the change.

13.1. Column budgets and energy transfer.

Let $\tau(n)$ count the positive divisors of n.

Lemma 13.1 (Finite-column transfer)**.** *Suppose $f - g$ is supported on at most J columns and $\|f-g\|_\infty \le B$. For (38), with $|a_c| \le A$, every $1 \le r < p$ satisfies*

$$\begin{aligned} |\Delta[f](r) - \Delta[g](r)| &\le AB \sum_{b:\, f_b \ne g_b} \big(\tau(p^2 - br) + \tau(p^2 + br)\big) \qquad (41) \\ &\le C_\varepsilon ABJp^\varepsilon \qquad (\varepsilon > 0). \end{aligned}$$

The constant is independent of the row and changed columns.

Proof. Each incidence uses a divisor of one of the displayed positive integers. Both integers are less than $2p^2$ because $b, r < p$. Triangle inequality over the changed columns proves the first bound.

For completeness, $\tau(n) \le C_\delta n^\delta$ for every $\delta > 0$. If a prime $\ell \ge 2^{1/\delta}$, then $a + 1 \le 2^a \le \ell^{\delta a}$. For each of the finitely many smaller primes,

$\sup_{a\ge 0}(a+1)\ell^{-\delta a}$ is finite. Multiplication over prime powers proves the divisor estimate. Take $\delta=\varepsilon/2$ in the first bound. $\square$

The same change has a controlled effect on energy without dropping the inherited interaction. Set $h(r)=\Delta[f](r)-\Delta[g](r)$ and $M_v[f]=\sum_r v(r)\Delta[f](r)^2$. Then

$$M_v[f]-M_v[g]=2\sum_r v(r)\Delta[g](r)h(r)+\sum_r v(r)h(r)^2, \tag{42}$$

$$\Big|\sqrt{M_v[f]}-\sqrt{M_v[g]}\Big|\le \|h\|_\infty\sqrt{\sum_r v(r)}.$$

The first formula is the field-change identity. The second is the reverse triangle inequality in the weighted square norm. Squaring a bound for h alone would miss the first term in (42).

13.2. **An adjustable smoothing width.** Smoothing has two costs. A wider transition changes more columns; a narrower transition has larger derivatives. Both costs must refer to the same register.

Choose fixed smooth functions χ on $[0,1]$ and u on the real line. Let χ vanish near zero, and let $u(t)=0$ for $t\le 0$, $u(t)=1$ for $t\ge 1$. For $1\le J\le p$ put

$$f_b^\sharp=\chi(b/p),\qquad B^{[J]}(\beta)=\chi(\beta)u\big(p(1-\beta)/J\big).$$

Extend $B^{[J]}$ by zero outside $[0,1]$. The sharp and smooth tables differ only on $p-J<b<p$. Differentiation and a change of variable over an interval of length J/p give

$$\|(B^{[J]})'\|_1\le C,\qquad \|(B^{[J]})''\|_1\le Cp/J. \tag{43}$$

Here and below the profile constants are fixed.

Proposition 13.2 (Balancing a boundary budget)**.** *Assume that the chosen modulus weights admit the smooth-row estimate*

$$|\Delta[B](r)|\le C_0\sqrt{p}\,\tau(r)\sqrt{\|B'\|_1\|B''\|_1} \tag{44}$$

for all the profiles $B=B^{[J]}$, uniformly in p,r,J. The notation $\Delta[B]$ means (38) with $f_b=B(b/p)$. If the modulus weights are uniformly bounded, then

$$|\Delta[f^\sharp](r)|\le C_\varepsilon Jp^\varepsilon+C_1\frac{p}{\sqrt{J}}\tau(r). \tag{45}$$

Choosing $J=\lceil p^{2/3}\rceil$ gives $|\Delta[f^\sharp](r)|\le C_\varepsilon p^{2/3+\varepsilon}$.

Proof. Lemma 13.1 bounds the difference between the sharp and smooth rows by $C_\varepsilon J p^\varepsilon$. Substituting (43) into (44) bounds the smooth row by $C_1 p\tau(r)/\sqrt{J}$. Add the two bounds and use $\tau(r) \le C_\varepsilon p^\varepsilon$. □

Thus this hypothesis supplies normalized row size $O_\varepsilon(p^{-1/3+\varepsilon})$ after division by p. The proposition isolates the reusable step: choose the transition width after estimating its actual column cost. The smooth-row bound is an explicit additional input; it is not a consequence of the incidence identity. In the motivating short-variable calculation that input uses completion of exponential sums and a classical Weil bound. Its analytic proof is not needed to apply the budget principle to another family with its own smooth estimate.

The exponents also locate the limit of this particular calculation. Writing $J = p^\theta$ gives the two powers θ and $1 - \theta/2$. Their maximum is least at $\theta = 2/3$. Improving the exponent requires a better estimate for at least one cost or cancellation between the retained terms.

13.3. **Reciprocal phase transfer.** A smooth-row calculation may contain an inverse modulo a large modulus. A finite identity transfers that inverse to a smaller one and records the correction. Write $e(x) = \exp(2\pi i x)$.

Proposition 13.3 (Reciprocal phase identity)**.** *For coprime integers $r, c > 1$ and any integer A,*

$$e(A\bar{r}/c) = e(-A\bar{c}/r)\, e(A/(rc)). \tag{46}$$

The bars denote inverses modulo the adjacent denominators.

Proof. Choose $1 \le u < c$ with $ru \equiv 1 \pmod{c}$ and put $v = (ru-1)/c$. Then $ru - cv = 1$, so $v \equiv -\bar{c} \pmod{r}$. Dividing by rc gives $u/c = v/r + 1/(rc)$. Exponentiation proves the identity; integer A makes the representatives immaterial. □

For $r = 4, c = 7$, the relation $4 \cdot 2 - 7 \cdot 1 = 1$ gives

$$e(2A/7) = e(-3A/4)e(A/28).$$

At $A = 121$ the two phases are $242/7$ and $-363/4 + 121/28$; they differ by 121, so their exponentials agree. The transferred inverse has modulus four. Its reciprocal correction $e(A/(rc))$ remains part of the field and may vary rapidly when A grows. The identity changes coordinates; any cancellation estimate must still control the resulting amplitudes and phases.

Problem 32 (A change of boundary cost)**.** Suppose a related calculation gives $|\Delta^\sharp| \le C(J + pJ^{-1/3})$, with $1 \le J \le p$. Choose the power of p for J that minimizes this allowance. What rate follows for $\Delta^\sharp/p$?

Problem 33 (The reciprocal phase correction)**.** In the reciprocal phase identity take $r = 5$, $c = 7$ and $A = 3$. Compute both modular inverses and verify the identity by comparing its arguments modulo one. Determine the absolute error caused by omitting the correction $e(A/(rc))$, where $e(x) = \exp(2\pi i x)$. For general coprime $r, c > 1$ and integer A, derive an upper allowance for that error in terms of $|A|/(rc)$.

14. Primitive shells and logarithmic transforms

Pairs $(1, 2), (2, 4), (3, 6)$ have the same ratio. A kernel unchanged by common dilation assigns them the same reading. In a square cutoff $1 \le a, b \le M$, the primitive pair $(1, 2)$ appears $\lfloor M/2 \rfloor$ times. Counting these repetitions is the starting point of a transform that separates primitive structure from its repeated observations.

In the arithmetic application, the kernel has a fixed modulus q, called its *carrier modulus*. The cutoff changes while q stays fixed. This is the arithmetic counterpart of keeping a field fixed while changing its observation. The general decomposition below needs only invariance under common dilation.

14.1. **Dilation-invariant kernels.** Fix a maximum cutoff M_0. Let a real kernel $R(a, b)$ on $1 \le a, b \le M_0$ satisfy

$$R(da, db) = R(a, b) \qquad \text{whenever } d \max(a, b) \le M_0.$$

Define its square prefixes and primitive shells by

$$\mathcal{A}(M) = \sum_{a,b \le M} R(a, b), \qquad P(m) = \sum_{\substack{\max(a,b)=m \\ (a,b)=1}} R(a, b), \qquad \mathcal{A}(0) = 0. \quad (47)$$

The inversion uses the Möbius coefficient $\mu(r)$. For a positive integer r, $\mu(r) = 0$ if a prime square divides r; otherwise $\mu(r) = (-1)^k$, where k is its number of distinct prime factors, with $\mu(1) = 1$. Thus $\mu(3) = -1$, $\mu(15) = 1$ and $\mu(9) = 0$.

Proposition 14.1 (Primitive-shell decomposition)**.** *For* $1 \le M \le M_0$,

$$\mathcal{A}(M) = \sum_{m \le M} \lfloor M/m \rfloor \, P(m), \tag{48}$$

$$\mathcal{A}(M) - \mathcal{A}(M-1) = \sum_{m \mid M} P(m),$$

$$P(m) = \sum_{d \mid m} \mu(d) \big(\mathcal{A}(m/d) - \mathcal{A}(m/d - 1) \big). \tag{49}$$

Proof. Every ordered pair is uniquely $d(a,b)$ with $(a,b) = 1$. A primitive pair with maximum m has $\lfloor M/m \rfloor$ admissible multiples. This proves the first identity. The increment of $\lfloor M/m \rfloor$ is one precisely when $m \mid M$, which proves the second. Finally $\sum_{d \mid n} \mu(d)$ is one at $n = 1$ and zero otherwise: for $n > 1$ its squarefree divisors give the expansion of $\prod_{\ell \mid n}(1-1)$. Applying this identity to the second formula gives (49). □

The inversion is classical Möbius inversion [1, Chapter 2]. The useful choice in an application is the kernel whose common dilations really preserve the reading. Signed values are allowed throughout; no positivity or asymptotic hypothesis enters the proposition.

Problem 34 (Repeated pairs)**.** Take $R(a,b) = 1$ and $M = 4$. Calculate the four primitive-shell coefficients and reconstruct $\mathcal{A}(4)$.

14.2. **The logarithmic prefix operation.** The factor $\lfloor M/m \rfloor$ counts every repetition of a primitive pair. A different weighting of those repetitions can select prime powers. The logarithmic operation below makes that change exactly, using the factorization identity for $\log n$.

Define the prime-power weight and its cumulative sum by

$$\Lambda(n) = \begin{cases} \log \ell, & n = \ell^j, \ \ell \text{ prime}, \ j \ge 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \Psi(x) = \sum_{n \le x} \Lambda(n).$$

Empty sums are zero. Unique factorization gives $\log n = \sum_{d \mid n} \Lambda(d)$, so Möbius inversion gives $\Lambda(n) = -\sum_{d \mid n} \mu(d) \log d$. Summing over $n \le x$ yields the finite identity

$$\Psi(x) = -\sum_{d \le x} \mu(d) \log d \, \lfloor x/d \rfloor \, . \tag{50}$$

These are the classical divisor identities for Λ [1, Chapter 2].

Apply the same coefficients to the square prefixes:

$$\mathcal{K}(N) = -\sum_{d\le N} \mu(d)\log d\,\mathcal{A}(\lfloor N/d\rfloor). \tag{51}$$

The calligraphic letter distinguishes this prefix reading from the mixed defect $K_q(f,g)$ used earlier.

Theorem 14.2 (Logarithmic shell transform)**.** *Under the hypotheses of Proposition 14.1, for every* $1 \le N \le M_0$*,*

$$\mathcal{K}(N) = \sum_{m\le N} P(m)\Psi(N/m). \tag{52}$$

Proof. Insert (48) into (51). The coefficient of $P(m)$ after exchanging the finite sums is

$$-\sum_{d\le N/m} \mu(d)\log d\left\lfloor \frac{N}{dm}\right\rfloor = \Psi(N/m),$$

by (50). The equality $\lfloor\lfloor N/d\rfloor/m\rfloor = \lfloor N/(dm)\rfloor$ ensures that no endpoint changes. □

This theorem composes gcd grouping with classical convolution. It applies to any kernel satisfying the dilation hypothesis. Its usefulness depends on the resulting shell coefficients and the estimates available for the shorter prefixes.

14.3. **A kernel supplied by clock boundaries.** The midpoint sawtooth supplies such a kernel. Recall ψ from Section 3, and put

$$s(h,k) = \sum_{j=1}^{k-1} \psi(j/k)\psi(hj/k), \qquad s(h,1) = 0.$$

Fix an odd prime p, let $q = p^2$, and restrict all indices to $1 \le a, b < p$. Define

$$R_q(a,b) = s(a\bar{q}, b) + s(b\bar{q}, a). \tag{53}$$

Every inverse uses the denominator of its own sum. The carrier q stays fixed in $\mathcal{A}_q(M)$, $P_q(m)$ and $\mathcal{K}_q(N)$.

Here is the connection to the boundary calculus. For $f_a(x) = 2\psi(ax)$,

$$K_{q,0}(f_a, f_b) = \frac{1}{q} - \frac{a/b + b/a}{3q^2} + \frac{4}{q}R_q(a,b). \tag{54}$$

To verify it first take $(a,b) = 1$. The product $f_a f_b$ has point correction one at zero and no other sampled boundary. At j/a, $1 \le j < a$, its value jump

is $-4\psi(bj/a)$; the other clock gives the symmetric terms. Their boundary sum is

$$\frac{4}{q}\sum_{j=1}^{a-1}\psi(bj/a)\psi(qj/a)+\frac{4}{q}\sum_{j=1}^{b-1}\psi(aj/b)\psi(qj/b)=\frac{4}{q}R_q(a,b),$$

after permuting residues by q. The slope-jump contributions, including the origin, are $-2q^{-2}(b/(6a)+a/(6b))$. This uses $\sum_{j=0}^{a-1}\widetilde{B}_2(qj/a)=1/(6a)$. The regular curvature is the constant $8ab$ and has zero Bernoulli integral. Theorem 4.1 now gives (54).

For a common divisor $d=(a,b)$, remove d from both clocks. Integer dilation preserves their continuous pairing and permutes the q-grid, since $(d,q)=1$. It also preserves the kernel:

$$s(dh,dk)=s(h,k).$$

Indeed, group the summation index by its residue modulo k and use

$$\sum_{j=0}^{d-1}\psi((x+j)/d)=\psi(x).$$

Inverses of q reduce consistently to smaller denominators. This proves both the general pair formula and the common-dilation hypothesis for R_q.

Thus primitive-shell grouping applies to the signed remainder already produced by the boundary decomposition. The classical Dedekind sum provides a compact notation for its boundary phases.

14.4. **A finite prime-counting example.** For $m>1$, pair the two orientations of a primitive collision. The unit residues in the second Dedekind term cancel under $k\mapsto m-k$, giving

$$P_q(m)=2\sum_{\substack{1\le k<m\\(k,m)=1}}s(m\bar{q},k). \tag{55}$$

Also $P_q(1)=0$. Directly from the defining sums,

$$s(1,2)=0,\qquad s(1,3)=\frac{1}{18},\qquad s(2,3)=-\frac{1}{18},\qquad s(1,4)=\frac{1}{8}.$$

Because $p^2\equiv 1 \pmod{12}$ for primes $p\ge 5$, these give

$$P_q(1)=P_q(2)=P_q(3)=0,\qquad P_q(4)=\frac{1}{9},\qquad P_q(5)=\frac{5}{36}, \tag{56}$$

with the last coefficient used when $p\ge 7$. For $m=4$ only $k=3$ contributes; for $m=5$ the nonzero terms are $k=3,4$.

Take $q = 121$ and $N = 10$. In (52), the first three shells vanish and all $m \geq 6$ have $10/m < 2$, so

$$\mathcal{K}_{121}(10) = \left(\frac{1}{9} + \frac{5}{36}\right) \log 2 = \frac{1}{4} \log 2.$$

This is a complete example of prime counting inside a transformed collision reading.

Primitive means coprime coordinates, not a prime shell index. For instance (55) gives $P_{49}(6) = -2/5$ and $P_{121}(6) = 2/5$, whereas $\Lambda(6) = 0$. The shell keeps its carrier dependence and signs. Those features matter when the identity is used for estimates.

Problem 35 (Logarithmic transform of a constant kernel)**.** For $R(a, b) = 1$, compute $\mathcal{K}(4)$ directly from the prefixes $\mathcal{A}(M) = M^2$ and then from the primitive shells.

Problem 36 (Signed primitive shells)**.** Let $R(a, b) = 2$ when $a = b$, let $R(a, b) = -1$ when $a = 2b$ or $b = 2a$, and let it be zero otherwise. Verify common-dilation invariance. Find every nonzero primitive-shell coefficient and the prefixes $\mathcal{A}(M)$ for $1 \leq M \leq 6$. Compute $\mathcal{K}(6)$ both from the logarithmic prefix definition and from the shell transform. Explain why some prefix increments vanish although the observed square grows.

15. Prescribed fields and quantitative recovery

An identity specifies the quantity to be estimated. Replacing it by a bound for every possible field can impose a much stronger requirement. For the matrix $\operatorname{diag}(T, -T)$, the vector $(1, 1)$ has quadratic reading zero although the operator norm is $|T|$. Large directions need not be the direction selected by the problem.

The distinction has an arithmetic instance in the kernel just constructed. It also appears when an exact inverse formula is used to recover individual prime-counting errors.

15.1. **An interior operator-norm obstruction.** Let $H = \lfloor \sqrt{p} \rfloor$ and choose real weights $0 \leq w_a \leq 1$ that equal one on $H \leq a \leq p - H$. These weights modify the kernel coefficients; they are distinct from the row masses of a register. Form the symmetric matrix

$$R_w = (w_a w_b R_q(a, b))_{1 \leq a, b < p}.$$

Its prescribed scalar reading is $\mathbf{1}^T R_w \mathbf{1} = \sum_{a,b<p} w_a w_b R_q(a,b)$. Let Π be orthogonal projection perpendicular to the constant vector. Removing that direction still leaves a large norm.

Proposition 15.1 (Interior norm bound)**.** *For every odd prime $p \geq 257$ and every weight table as above,*

$$\|\Pi R_w \Pi\|_{\mathrm{op}} \geq p/64. \tag{57}$$

Proof. Choose the largest odd integer $r \leq \sqrt{p}/2$ and put

$$a = r^2, \qquad b = (p+r)/2, \qquad t = 2(p-r).$$

Then $p^2 = a + tb$ and $(a,b) = (t,a) = 1$. Both indices lie in the plateau. To see the lower bound on a, write $y = \sqrt{p} \geq 16$. Then $r > y/2 - 2$ gives $a > p/4 - 2y + 4 \geq y$. The other bounds follow from $a \leq p/4$ and $p/2 \leq b \leq p/2 + y/4$.

Reduction of $p^2 = a + tb$ modulo each denominator gives

$$R_q(a,b) = s(1,b) + s(t,a).$$

Here $s(\bar{t},a) = s(t,a)$ by permuting the defining sum. Cauchy–Schwarz in that sum gives $|s(t,a)| \leq s(1,a)$. Summing the squares of $j/k - 1/2$ yields

$$s(1,k) = \frac{k}{12} - \frac{1}{4} + \frac{1}{6k}.$$

Since $b - a \geq p/4$, it follows that

$$R_q(a,b) \geq \frac{b-a}{12} + \frac{1}{6}(1/b - 1/a) \geq \frac{p}{48} - \frac{1}{6} \geq \frac{p}{64}.$$

The diagonal entries of R_w vanish and $w_a = w_b = 1$. For $v = (e_a - e_b)/\sqrt{2}$, $\Pi v = v$, $\|v\| = 1$, and $v^T \Pi R_w \Pi v = -R_q(a,b)$. The definition of the operator norm proves the claim. □

Thus a uniform $O_\varepsilon(p^{1/2+\varepsilon})$ norm bound, for every $\varepsilon > 0$, is unavailable even on the mean-zero subspace. This does not decide the prescribed scalar sum. The lesson is to retain the source vector when choosing an estimate: an operator bound must pay for directions that the scalar may never use.

15.2. **Exact recovery and stability.** The logarithmic transform has a different issue. It can reconstruct a quantity exactly while enlarging a termwise

error allowance. Keep the carrier $q = p^2$ fixed. Since the first nonzero shell is $P_q(4) = 1/9$, the prime-counting identity at $N = 4n < p$ gives

$$\Psi(n) = 9\mathcal{K}_q(4n) - 9\sum_{m=5}^{4n} P_q(m)\Psi(4n/m). \tag{58}$$

For integer $n \geq 2$, every argument on the right has integer part less than n. The initial prime count is $\Psi(x) = 0$ for $x < 2$. Hence the formula is a finite triangular recovery, despite $P_q(1) = 0$ preventing an ordinary Dirichlet inverse.

To examine errors, retain the complete linear term:

$$F_q(N) = \mathcal{K}_q(N) - N\sum_{m\leq N}\frac{P_q(m)}{m}, \qquad E(x) = \Psi(x) - x.$$

Subtracting that term from (52) gives

$$F_q(N) = \sum_{m\leq N} P_q(m)E(N/m), \tag{59}$$

$$E(n) = 9F_q(4n) - 9\sum_{m=5}^{4n} P_q(m)E(4n/m). \tag{60}$$

These formulas include arguments between one and two, where $E(x) = -x$.

Suppose an induction tries to propagate the allowance $|E(x)| \leq Cx^\alpha$ by taking absolute values in (60). The $m = 5$ term alone has coefficient

$$9|P_q(5)|(4/5)^\alpha = \frac{5}{4}(4/5)^\alpha = (5/4)^{1-\alpha}. \tag{61}$$

For $\alpha < 1$ this exceeds one. At square-root scale it is $\sqrt{5}/2$. Thus that absolute induction does not contract. The signed recurrence remains intact; the calculation identifies the information discarded by the attempted estimate.

15.3. **Additional estimates for the arithmetic application.** The transform uses shorter prefixes at the same carrier. At $N = p-1$, the coefficient of $d = 1$ in (51) vanishes because $\log 1 = 0$. Every remaining prefix therefore has cutoff at most $(p-1)/2$. A bound on $\mathcal{A}_{p^2}(p-1)$ alone supplies neither those readings nor their logarithmically weighted sum. Replacing p by a smaller prime also changes the kernel.

A quantitative prime-distribution application needs estimates for these fixed-carrier prefixes and control of the signed recovery in (60). Those estimates are open for the collision family considered here. The proved tools

are the finite decomposition, the transform and the recovery identity. Their use has made the missing information explicit.

The same discipline applies to the register moment. Its finite pair formula retains all signs and observed intersections; an improved rate requires an estimate for their joined sum. The smoothing proposition gives the stated one-third normalized rate under its explicit analytic hypothesis. It supplies no stronger rate merely by changing the width.

Problem 37 (A quantitative recovery test)**.** A triangular recurrence has first nonzero coefficient $P(a) = c \neq 0$ and a later coefficient $P(b) = d$, with $b > a$. What coefficient does the b term contribute to a termwise allowance Cx^α after solving for the value at n using the observation at an? Apply your answer to $a = 4$, $b = 5$, $c = 1/9$, $d = 5/36$.

Problem 38 (A contracting recovery)**.** Let $E(x) = 0$ for $0 \leq x < 1$ and define $F(x) = E(x/2) - \frac{1}{4}E(x/4)$. Suppose $|F(x)| \leq C\sqrt{x}$ for every real $x \geq 2$. Recover $E(n)$, for real $n \geq 1$, as a finite sum of shorter observations of F, and derive an explicit square-root bound. Compute the contraction factor and compare it with the factor in Problem 37.

Part 4

Arithmetic energy at growing scales

The preceding parts give operations for an exact finite account. A growing arithmetic field also requires control of the dependence on scale. The two applications here make that dependence concrete.

Carry tables supply a periodic clock field. Moving its sampling grid changes the simultaneous boundary readings, and the half-cell calculation follows that change through to a limit. Tapered divisor families change both their source and their activation cutoff. Completed-cycle screening controls part of their interaction; the whole-field calculation retains the remaining tail, parent shell and combined resets. Local invariant readings then measure energy that later additions must leave behind.

16. Carry tables and clock families

In base three, the equal-digit words $00, 11, 22$ have numerical values $0, 4, 8$. They pick out three positions among the nine possible two-digit words. More generally, a repeated digit dd in base p has value $(p+1)d$. The carry count defined below tests these selected positions at each integer address a. A floor difference counts the integers crossed between na/q and $(n+1)a/q$. Centering removes the term a/p so that the remaining fluctuation can be compared with a clock field.

Let p be an odd prime, $N = p-1$ and $q = p^2$. The equal-digit base-p words occupy the positions $G_p = \{(p+1)d : 0 \le d < p\}$ on a circle of q positions. Their carry count and centered reading are

$$C_p(a) = \sum_{n \in G_p} \left(\left\lfloor \frac{(n+1)a}{q} \right\rfloor - \left\lfloor \frac{na}{q} \right\rfloor \right), \qquad F_p(a) = C_p(a) - \frac{a}{p}.$$

Write $E_p = \sum_{a=0}^{q-1} F_p(a)^2$ and $B_N(x) = 2\sum_{j=1}^{N} \psi(jx)$. For $p = 3$ and $a = 1$, the three floor differences are $0, 0, 1$, so $C_3(1) = 1$ and $F_3(1) = 2/3$. The next lemma identifies every centered table reading with a reading of B_2, after permuting the nine addresses. It therefore transfers the entire squared table, not just its average, to the clock calculation.

Lemma 16.1 (Carry representation)**.** *If $[z]_q$ is the least residue modulo q, then*

$$F_p([(1-p)A]_q) = B_N(A/q), \qquad E_p = \sum_{A=0}^{q-1} B_N(A/q)^2.$$

Proof. Expanding each floor as its argument minus its fractional part cancels the allowance a/p. At $a = [(1-p)A]_q$, the congruences $(1-p)(p+1)d \equiv d$

$\pmod q$ and $(1-p)((p+1)d+1) \equiv d+1-p \pmod q$ give

$$F_p(a) = \sum_{j=1}^{p-1} \big(\{jA/q\} - \{-jA/q\}\big) = 2\sum_{j=1}^{p-1} \psi(jA/q).$$

The equality includes integer arguments. Multiplication by $1-p$ is a permutation modulo q, proving the energy identity. □

At a reduced boundary a/k, the clocks whose frequencies are multiples of k reset together. There are $\lfloor N/k \rfloor$ of them. Grouping the other frequencies into blocks of length k cancels each complete block; only the final incomplete block contributes to the inherited reading. The next notation records the number of complete blocks, the remaining length and that remaining reading. Each reduced fraction is counted once by requiring a and k to be coprime.

For $2 \le k \le N$, set

$$M_k = \lfloor N/k \rfloor, \quad s_k = N - kM_k, \quad T_{k,s}(a) = 2\sum_{j=1}^{s} \psi(ja/k), \quad H_N = \sum_{j=1}^{N} \frac{1}{j}.$$

Let $\mathcal{U}_k = \{1 \le a < k : (a,k) = 1\}$, and put

$$\mathcal{A}_p = \sum_{k=2}^{N} M_k \sum_{a \in \mathcal{U}_k} T_{k,s_k}(a)\psi(p^2 a/k).$$

Proposition 16.2 (Carry boundary identity)**.** *The carry sampling defect satisfies*

$$D_{p^2}[B_{p-1}] = \frac{N^2}{p^2} + \frac{4\mathcal{A}_p}{p^2} - \frac{NH_N}{3p^3}. \tag{62}$$

Proof. The common reset at zero has $C_0 = N$ and $U_0 = 0$. At a reduced interior boundary a/k, the resetting clocks are the multiples of k, so $C_{a/k} = M_k$. The nonreset readings in each complete block of k clocks sum to zero, leaving $U_{a/k} = T_{k,s_k}(a)$. No interior boundary lies on the p^2 grid, because $k < p$ and $(k, p^2) = 1$. The first two terms of (7) are therefore N^2/p^2 and $4\mathcal{A}_p/p^2$.

The slope is $V = N(N+1) = pN$. To evaluate the last term, use

$$\sum_{b\in\mathcal{B}} C_b \widetilde{B}_2(p^2 b) = \frac{N}{6} + \sum_{k=2}^{N} M_k \sum_{a\in\mathcal{U}_k} \widetilde{B}_2(p^2 a/k)$$
$$= \sum_{j=1}^{N}\sum_{a=0}^{j-1} \widetilde{B}_2(p^2 a/j) = \sum_{j=1}^{N} \frac{1}{6j} = \frac{H_N}{6}.$$

The middle identity counts each reset once for each clock that resets there. Multiplication by p^2 permutes residues modulo every $j < p$. Finally,

$$\sum_{a=0}^{j-1} \widetilde{B}_2(a/j) = \frac{1}{6j},$$

obtained by summing the quadratic polynomial using the sums of integers and their squares. Substitution in (7) gives the harmonic term. □

The continuous energy can also be read from pairwise clock collisions. The Fourier coefficients of ψ are $-1/(2\pi i n)$ for $n \neq 0$. The frequencies common to $\psi(jx)$ and $\psi(kx)$ give, by Parseval,

$$\int_0^1 \psi(jx)\psi(kx)\,dx = \frac{(j,k)^2}{12jk}, \qquad \int_0^1 B_N^2\,dx = \frac{1}{3}\sum_{j,k\le N} \frac{(j,k)^2}{jk}.$$

Together with Lemma 16.1, this recovers both sides of the carry-energy comparison from the same finite family.

For $p = 3$, $\mathcal{A}_3 = 0$ and $H_2 = 3/2$, giving $D_9[B_2] = 4/9 - 1/27 = 11/27$. In general, the established unit-sampling theorem for this carry family [5] supplies the separate estimate

$$D_{p^2}[B_{p-1}] = 1 + O\big(p^{-1/458696}\log^4(2p)\big)\,.$$

Equation (62) identifies its residual as $4\mathcal{A}_p/p^2 - NH_N/(3p^3)$. The boundary decomposition does not replace the signed estimate proving that this residual tends to zero. The finite common-reset contribution is $(p-1)^2/p^2$.

Problem 39 (A base-five carry table)**.** Take $p = 5$, so $q = 25$ and $N = 4$. For $A = 0, 1, 5, 12$, compute $a = [(1-p)A]_{25}$ and evaluate $C_5(a)$ directly from the floor sum. Compare $F_5(a)$ with $B_4(A/25)$. Explain why this address change preserves the complete squared table sum, even though the two formulas need not agree at the same address.

17. Half-cell carry grids

At the origin all $p-1$ clocks reset together. A half-cell shift misses that point and samples the reset at one half, where only the even-frequency clocks reset. The number of resetting clocks is halved, so their squared amplitude is quartered. This identifies the candidate change in the limit. The rest of the section calculates the other boundary terms and controls their combined contribution.

Retain the field B_N, with $N = p-1$, but sample it at $(a+1/2)/p^2$. This changes the sampling rule while leaving the continuous energy fixed. It is a shifted observation of the clock field; no new interpretation of these points as integer carry inputs is required.

Put $M = N/2$ and define the shifted interior source

$$\mathcal{A}_p^{\mathrm{half}} = \sum_{k=2}^{N} M_k \sum_{a \in \mathcal{U}_k} T_{k,s_k}(a)\psi(p^2 a/k - 1/2).$$

The location $1/2$ is now sampled. Exactly the even-frequency clocks reset there. All the odd-frequency clocks have reading zero there as well, since $\psi(j/2) = 0$ for odd j. Consequently $C_{1/2} = M$, $U_{1/2} = 0$, and $B_N(1/2) = 0$.

Proposition 17.1 (Half-cell carry identity)**.** *For every odd prime p, with $N = p-1$ and $M = N/2$,*

$$D_{p^2,1/2}[B_N] = \frac{M^2}{p^2} + \frac{4\mathcal{A}_p^{\mathrm{half}}}{p^2} - \frac{N(3H_M - 2H_N)}{12p^3}. \tag{63}$$

The first term is the complete sampled point correction. The only sampled reset is at $1/2$.

Proof. For a reduced boundary a/k, $k \le N$, membership in the grid requires $p^2 a/k - 1/2 \in \mathbb{Z}$. Since $(k, p^2) = 1$, this forces $k \mid 2$. Zero is not sampled, whereas $1/2$ is sampled because p^2 is odd. Thus the point correction is M^2/p^2. The value-jump term is $4\mathcal{A}_p^{\mathrm{half}}/p^2$ by the same boundary readings used in Proposition 16.2.

For the slope term, count boundaries by the clocks resetting there:

$$\sum_b C_b \widetilde{B}_2(p^2 b - 1/2) = \sum_{j=1}^{N} \sum_{a=0}^{j-1} \widetilde{B}_2(p^2 a/j - 1/2)$$
$$= \sum_{j=1}^{N} \frac{\widetilde{B}_2(j/2)}{j} = \frac{H_M}{8} - \frac{H_N}{12}.$$

The second equality uses the residue permutation modulo j and

$$\sum_{a=0}^{j-1} \widetilde{B}_2(a/j + t) = \frac{1}{j} \widetilde{B}_2(jt).$$

This multiplication identity follows by summing the absolutely convergent Fourier series of $\widetilde{B}_2$: only frequencies divisible by j remain. For the final equality, even j have $\widetilde{B}_2(j/2) = 1/6$ and odd j have $\widetilde{B}_2(j/2) = -1/12$. The even reciprocal sum is $H_M/2$. Substituting $V = pN$ into (7) proves the formula. □

For $p = 3$ the nine readings on the half-cell grid are

$$-\frac{5}{3},\ -1,\ -\frac{1}{3},\ \frac{1}{3},\ 0,\ -\frac{1}{3},\ \frac{1}{3},\ 1,\ \frac{5}{3}.$$

Their squares sum to eight. The continuous energy is one, so the defect is $1 - 8/9 = 1/9$, agreeing with the boundary calculation. In this example both the shifted interior source and the harmonic correction vanish.

17.1. **Residual criterion.** The point correction satisfies

$$\frac{M^2}{p^2} = \frac{(p-1)^2}{4p^2} \longrightarrow \frac{1}{4},$$

and the harmonic term is $O(p^{-2} \log(2p))$. As p tends to infinity through odd primes,

$$D_{p^2,1/2}[B_{p-1}] \longrightarrow \frac{1}{4} \quad \Longleftrightarrow \quad \mathcal{A}_p^{\text{half}} = o(p^2). \tag{64}$$

This equivalence distinguishes a limit for the sampled collision amplitude from a limit for the complete energy defect.

To compare the new source with the unshifted one, define for positive integers N, Q

$$\mathcal{A}_N(Q) = \sum_{k=2}^{N} \lfloor N/k \rfloor \sum_{a \in \mathcal{U}_k} T_{k, N \bmod k}(a) \psi(Qa/k).$$

There is no coprimality hypothesis in this definition. The original source is $\mathcal{A}_p = \mathcal{A}_{p-1}(p^2)$.

Proposition 17.2 (Shift and parity reduction)**.** *For $N = p-1$ and $q = p^2$,*

$$\mathcal{A}_p^{\text{half}} = \mathcal{A}_N(2q) - \mathcal{A}_N(q). \tag{65}$$

For arbitrary positive integers N, Q, put

$$R_k(u, v) = \sum_{a=0}^{k-1} \psi(ua/k)\psi(va/k).$$

Then

$$\mathcal{A}_N(Q) = 2 \sum_{l, j \le N} R_l(j, Q), \tag{66}$$

and the even moduli obey

$$R_{2k}(j, 2q) = \begin{cases} 2R_k(j/2, q), & j \text{ even}, \\ R_k(j, q), & j \text{ odd}. \end{cases} \tag{67}$$

Proof. The integer-zero convention gives the pointwise identity

$$\psi(x - 1/2) = \psi(2x) - \psi(x).$$

It follows by considering $0 < \{x\} < 1/2$ and $1/2 < \{x\} < 1$; at integers and half-integers both sides vanish. Substitution proves (65).

For (66), complete periods of $\psi(ja/k)$ sum to zero, so $T_{k, N \bmod k}(a)$ can be replaced by $2\sum_{j=1}^{N} \psi(ja/k)$ for $(a, k) = 1$. Expand $\lfloor N/k \rfloor$ as the number of multiples of k through N. Reducing each fraction a/l, $0 \le a < l$, counts every reduced boundary with precisely that multiplicity. The boundary at zero contributes zero. This yields the full sum over l, j without requiring an inverse of Q.

For (67), group the residues modulo $2k$ as $a = b$ and $a = b + k$. The second factor has the same value $\psi(qb/k)$ in both terms. If j is even, the first factors are equal. If j is odd, their sum is

$$\psi(jb/(2k)) + \psi(jb/(2k) + 1/2) = \psi(jb/k).$$

This is the same multiplication identity for ψ, valid at its midpoint readings. The two cases give the asserted reduction. $\square$

The same shift also has a direct grid interpretation:

$$D_{q,1/2}[B_N] = 2D_{2q}[B_N] - D_q[B_N].$$

It follows from (12). This relation is exact for each fixed N. It does not supply an estimate for D_{2q} along the growing family.

17.2. **Parity-restricted source.** The unshifted estimate controls $\mathcal{A}_N(q)$ for $N = p-1$, $q = p^2$. Its count-source transformation uses the inverse of q modulo every $l \le N$. The replacement $q \mapsto 2q$ is not permitted at even moduli. For instance, multiplication by 25 permutes the residues modulo four, while multiplication by 50 reaches only the even residues. An inverse exists for the first map and not for the second. Formula (67) removes that algebraic obstruction, and leaves three parity-restricted sums. Their cancellation can be transferred by retaining parity through the full reciprocity descent.

Recall the Dedekind sum from Section 14. Since $\psi(0) = 0$, its equivalent form with the zero term included is $s(h,k) = \sum_{a=0}^{k-1} \psi(a/k)\psi(ha/k)$, with $s(h,1) = 0$. The argument h need not be a unit. At odd l, $2q$ is invertible; at every $k \le M$, q is invertible. Hence the remaining source is exactly

$$\begin{aligned}\mathcal{A}_N(2q) = 2 \sum_{\substack{l \le N \\ l \text{ odd}}} \sum_{j=1}^{N} s(j\overline{2q}, l) \\ + 4 \sum_{k=1}^{M} \sum_{r=1}^{M} s(r\bar{q}, k) + 2 \sum_{k=1}^{M} \sum_{r=1}^{M} s((2r-1)\bar{q}, k).\end{aligned} \tag{68}$$

The modulus-one terms are understood as zero. The bars denote inverses in the modulus of their summand. This formula follows by splitting (66) into odd and even l and then using (67).

The finite identities above identify the new source created by the shift. The next lemma supplies the additional analytic transfer needed for its limit. The unit-limit theorem alone does not supply this step. Readers interested first in using the calculus may continue at Theorem 17.4 and its interpretation, then return to the transfer proof with the cited analytic source available.

17.3. **Parity transfer.** Parity restricts the source to a fixed number of arithmetic progressions in the reciprocity calculation. The following estimate makes that transfer explicit. Its analytic ingredients are the reciprocal-phase, completion and stopped-descent estimates in [5, Sections 4–12].

The two cutoff variables X, Y describe a rectangle of indices, allowing its sides to end at different places. Uniformity in both matters because splitting into even and odd indices changes those endpoints. The proof first converts clock correlations into modular interval counts. Finite Fourier sums retain the restriction to odd indices while completing the counts to whole residue systems. Reciprocity then reduces the modular pairs to smaller ones; the restriction is followed through that reduction. The required saving is below p^2. Once divided by the sampling scale p^2, the bound in the lemma tends to zero.

Lemma 17.3 (Parity-restricted rectangles)**.** *Let p be an odd prime, $1 \le X, Y < p$, and put $\eta = 1/458696$. Each of the following sums is bounded by*

$$O(p^{2-\eta}\log^4(2p)),$$

uniformly in X, Y:

$$2\sum_{l\le X}\sum_{j\le Y} s(j\overline{p^2}, l),$$

$$2\sum_{\substack{l\le X\\ l \text{ odd}}}\sum_{j\le Y} s(j\overline{2p^2}, l),$$

$$2\sum_{l\le X}\sum_{\substack{j\le Y\\ j \text{ odd}}} s(j\overline{p^2}, l).$$

Each inverse is taken in its summand's modulus; modulus-one terms are zero. The implied constant is absolute.

Proof. Write $l = rk$, $j = ra$, $(a, k) = 1$, and set $U_r = \lfloor X/r\rfloor$, $L_r = \lfloor Y/r\rfloor$. The distribution identity for ψ gives $s(j\bar{q}, l) = s(a\bar{q}, k)$. Pairing opposite units therefore expresses each rectangle as a sum of $s(h, k)W_{r,k}(qh)$ over $k \le U_r$ and $h \in \mathcal{U}_k$. Here $q = p^2$ or $2p^2$. For the first rectangle the source is

$$\Delta_L(y; k) = \mathbf{1}_{1\le y \bmod k\le L \bmod k} - \mathbf{1}_{1\le -y \bmod k\le L \bmod k}, \qquad L = L_r.$$

For the second it is the same source with r, k both odd. For the third, r is odd. At even k every unit numerator is odd, so the source is again Δ_L. At

odd k it is

$$W_L(y;k) = \#\{1 \le a \le L : a \text{ odd},\ a \equiv y\ (k)\} \\ - \#\{1 \le a \le L : a \text{ odd},\ a \equiv -y\ (k)\}.$$

All these sources are odd and bounded by one in absolute value. The restriction on r only removes terms from the absolute sums over source indices used below.

We first retain the endpoints of the new odd source. Put $R = \lceil L/2 \rceil$, $m = R \bmod k$, $a_2 = (k+1)/2$, and $I_m(z) = \mathbf{1}_{1 \le z \bmod k \le m}$. Counting $a = 2v - 1$ gives

$$W_L(y;k) = I_m(a_2(y+1)) - I_m(a_2(1-y)). \tag{69}$$

For $b(x) = \{x\} - 1/2$, with its actual value $-1/2$ at integers,

$$I_m(z) = \frac{m}{k} - b((z-1)/k) + b((z-m-1)/k).$$

This identity holds for every integer z, including zero residues. The two constants m/k cancel in (69). In its four b terms, m can be replaced by R, since the difference is a multiple of k. On either parity class of y, the four phases have the form

$$\theta + \frac{\sigma y/2 + v}{k}, \qquad \sigma \in \{-1, 1\}, \quad v \in \{-1/2, -R - 1/2\}, \tag{70}$$

where $\theta = (\sigma y + 1)/2 \bmod 1$ is constant on that class. In particular $|v| \le p/2$. The ordinary source has the literal formula

$$\Delta_L(y;k) = \frac{1}{k} - 2b(y/k) + b((y-L-1)/k) + b((y+L)/k) \quad (y \not\equiv 0 \pmod{k}).$$

Its shifts have absolute value at most p.

Completion at fixed k also survives. For centered nonzero frequencies, $|\widehat{I}_m(n)| \ll 1/|n|$. Expanding the two interval functions in (69) gives Kloosterman sums with second frequencies $\pm n a_2 q$ and unit-modulus constant factors. Since $(a_2 q, k) = 1$, their gcd factors are the same as those for n. The zero frequency cancels. Thus, for a weight f with $f(0) = 0$ and cyclic variation V_k, the completion estimate remains

$$\left| \sum_{c \in \mathcal{U}_k} f(c) \sum_{r \le X/k} W_{r,k}(q\bar{c}) \right| \ll \frac{pV_k}{k} \tau(k) \big(\sqrt{k} \log^2(2k) + \log(2k) \big). \tag{71}$$

Any restrictions on r are retained. This argument uses the Fourier coefficients before the affine permutation. It does not assert small variation of W_L in its original residue order.

Inversion and reflection now give the same nearest-remainder descent of $s(c,k)$ as for the ordinary prefix. A parity condition on k is fixed during completion. The stopped-cost weight therefore keeps the bound $V_k \le 8(k/H)^2$ from the complete descent. The other estimates sum reciprocal phases along $k = ct + a$, or along $c = dt + a$, $k = mc \pm d$. In each case an integer representative of $\bar{c}$ is affine in t. For instance, with $fa \equiv 1 \pmod{c}$,

$$h = (1 - fk)/c$$

is such a representative in the first case. In the second one may use $h = (\varepsilon fk - \varepsilon m)/d$, where $fa \equiv 1 \pmod{d}$ and $k = mc + \varepsilon d$. The modulus and representative therefore have fixed parities on at most two progressions in t.

On each progression the phases in (70) are reciprocal phases with coefficient $q/2$ or $-q/2$; ordinary sources have coefficient q. Their magnitudes are comparable to p^2/c^2 or p^2/d^2. The endpoint shifts are at most $p/2$ for coefficient $p^2/2$, and at most p for coefficients $p^2, 2p^2$. The strict inequalities $2c < k < p$ and $4d < k < p$ keep these shifts smaller than the leading amplitudes by fixed factors. Replacing t by $2v + \epsilon$ in a reciprocal phase only changes its amplitude and denominator shift by fixed factors. The reciprocal-phase bound $O(|Z|^{1/3}\log(2p))$ and the variation estimates for its Dedekind weight therefore apply on every resulting interval.

With $\ell = \log(2p)$, the first small pairs, rational correction, first crossing and complete stopped cost consequently have bounds

$$\begin{gathered} p^{5/3}H^{1/3}\ell^3, \qquad p^{5/2}H^{-1}\ell^3, \\ p^{5/3}H^{1/3}\ell^3 + pH\ell^4, \qquad p^{7/2}H^{-2}\ell^3, \end{gathered} \tag{72}$$

respectively. These are the four estimates of the original descent. The preceding completion and progression checks account for each change to its source. Distinct cutoffs U_r, L_r do not alter them: U_r bounds the interval length, while L_r supplies only the retained shifts, with the bounds just stated.

It remains to check the full terminal population, where a restriction cannot be discarded from a signed sum. Include the crossing step in its prefix matrix $\left(\begin{smallmatrix} A & B \\ C & E \end{smallmatrix}\right)$, with determinant δ. Its terminal variables obey

$$u > H, \quad 1 \le e \le H, \quad 2e < u, \quad (u,e) = 1, \qquad k = Au + Be \le U_r.$$

Put $f = \bar{u} \bmod e$, $v_0 = (1 - fu)/e$, using $f = 0$ at $e = 1$, and $h = Av_0 - Bf$. The determinant identity gives $\bar{c} \equiv \delta h \pmod{k}$. Since every source is odd, $\delta W_{r,k}(q\bar{c}) = W_{r,k}(qh)$. The two orientations thus still join into the full prefix population

$$(A, B) = 1, \quad |B| \geq 2, \quad B/A \in (-\beta, \alpha), \quad \alpha = (\sqrt{5} - 1)/2, \quad \beta = 1 - \alpha.$$

The parity of k remains a condition on this population, and the parity of h is used only to expand its already joined source. Let $G_{r,K}(u, e)$ be the sum of $W_{r,k}(qh)$ over this population with $K \leq A < 2K$, the cutoff $k \leq U_r$ and the retained modulus parity. The terminal block is $\mathcal{B}_r(K; H) = 2\sum_{u,e} s(e, u)G_{r,K}(u, e)$, over the terminal domain displayed above.

For a small dyadic block $K \leq A < 2K$, set $u = et + a$ at fixed matrix and residue a. Both k and $h = \delta(A(1 - fu)/e - Bf)$, before the orientation fold, are affine in t. Splitting its parity again costs a fixed number of progressions. The original small-block estimate therefore reads

$$|\mathcal{B}_r(K; H)| \ll p^{2/3} U_r H^{1/3} K \ell^3. \tag{73}$$

Here the Dedekind weight's endpoint size and variation are $O(U_r/(Ae) + 1 + |s(a, e)|)$, exactly as in the unmodified descent.

For a large block, hold u, e, B fixed and remove coprimality by $\mathbf{1}_{(A,B)=1} = \sum_{d|A,\ d|B} \mu(d)$, $A = da$. The remaining inequalities give one interval in A. Since $k = Au + Be$ and $h = Av_0 - Bf$ are affine in A, all parity restrictions leave at most two progressions of step at most two in a. On the odd-modulus branch an even d contributes nothing. The map $(A, B) \mapsto (k, h)$ has determinant -1, so $(k, h) = (A, B) < k$. Thus the ordinary literal formula still applies after coprimality is removed: q is coprime to every allowed k. The affine interval formula needs no such exclusion and remains exact on these nonprimitive pairs as well.

For any of the retained phases put $q' = q$ or $q' = \pm q/2$, as appropriate. As a real identity,

$$\frac{q'h + v}{k} = \gamma - \frac{Z_v}{A + \zeta}, \qquad Z_v = \frac{q'B}{u^2} - \frac{v}{u}, \quad \zeta = \frac{eB}{u}, \quad \gamma = -\frac{q'f}{e} + \frac{q'}{eu}.$$

The same interval inequalities give $|Z_v| \asymp p^2|B|/u^2$ and $A + \zeta \asymp K$, with absolute comparison constants. In particular, the shift ratio is at most $2/(K|B|)$, since $u < 2p/K$ and $|B| \geq 2$. Take $H = \lfloor p^{11/14} \rfloor$, $K > p^{1/28}$, and $D = J = \lfloor K^{1/100} \rfloor$. On $d \leq D$ put $M = K/d$. For frequencies $1 \leq j \leq J$, the phase derivatives on each parity progression have sizes comparable to

$$TM^{-a} \quad (2 \leq a \leq 14), \qquad T = j|Z_v|/K, \qquad K \ll T \ll K^{12+1/100}.$$

Thus $M^{1/2} \le T \le M^{13}$ once K exceeds an absolute constant. The finitely many smaller dyadic scales are absorbed by increasing the implied constant. The fixed derivative-window estimate from [5, Section 11] supplies the saving $\kappa = (2^{14} - 2)^{-1}$ on each progression. Erdős–Turán summation includes integer phase hits and gives

$$O\big(M/J + M^{1-\kappa}\log(2J) + 1/(du)\big)$$

for each B, d; the last term retains the ordinary source's $1/k$. There are $O(K/d)$ possible multiples B of d. Summing them and then bounding $d > D$ by its population gives

$$\begin{aligned}|G_{r,K}(u,e)| &\ll K^2/J + K^{2-\kappa}\log(2K)\\ &\quad + K/u + K^2/D + K\log(2K)\\ &\ll K^{2-\kappa}\log(2K).\end{aligned}$$

No parity restriction has been removed from the joined source.

The terminal Dedekind weights have total absolute mass $O((U_r/K)^2\ell^2)$. Use (73) below $K_0 = p^{1/28}$ and the preceding saving above it. The bounds $\sum_r U_r \ll p\ell$ and $\sum_r U_r^2 \ll p^2$ give a total terminal cost

$$O\big(p^{55/28}\ell^4 + p^{2-\kappa/28}\ell^3\big)\,.$$

At the chosen H, (72) costs $O(p^{27/14}\ell^3)$. Since $\eta = \kappa/28$, all contributions satisfy the asserted estimate. □

Theorem 17.4 (Half-cell carry limit)**.** *As p tends to infinity through odd primes,*

$$D_{p^2,1/2}[B_{p-1}] = \frac{1}{4} + O\big(p^{-1/458696}\log^4(2p)\big)\,.$$

Proof. The three terms of (68) are respectively the second rectangle in Lemma 17.3, twice its first rectangle with $X = Y = M$, and its third with $X = M$, $Y = N$. Thus $\mathcal{A}_N(2p^2) = O(p^{2-\eta}\log^4(2p))$. The first rectangle with $X = Y = N$ supplies the same bound for $\mathcal{A}_N(p^2)$. Equation (65) gives that bound for $\mathcal{A}_p^{\mathrm{half}}$. Insert it in (63); $M^2/p^2 = 1/4 + O(1/p)$ and the harmonic term is $O(p^{-2}\log(2p))$. □

The grid shift changes which combined reset is sampled. Its surviving amplitude falls from N to $N/2$. The two limits, one and one quarter, record the square of that change after the interior source has been estimated. The boundary calculation identifies the limit's contribution; the parity transfer controls everything that remains.

17.4. **Finite readings of the shifted field.** The boundary term and the full defect approach their limits in different ways. The following values separate the sampled point contribution M^2/p^2 from the signed source $4\mathcal{A}_p^{\mathrm{half}}/p^2$. They are rounded displays of rational calculations made with *nfield* [8], software for exact arithmetic calculations.

Each row can be reconstructed by evaluating B_N at the p^2 half-cell sample points, averaging their squares, and subtracting that average from the continuous Gram sum in Section 16. The boundary formula (63) gives an independent calculation of the same defect.

p	M^2/p^2	$D_{p^2,1/2}[B_N]$	$4\mathcal{A}_p^{\mathrm{half}}/p^2$
3	0.111111	0.111111	0.000000
5	0.160000	0.123556	−0.035556
11	0.206612	0.101023	−0.104967
31	0.234131	0.232033	−0.001934
53	0.240655	0.259267	0.018684
61	0.241870	0.290206	0.048394
97	0.244872	0.206508	−0.038337
101	0.245074	0.258963	0.013914

The sampled point correction increases toward one quarter. The full defect fluctuates on both sides of that value in the displayed range. For example, the boundary contributions at $p = 53$ and $p = 97$ are already close, but their signed sources have opposite signs. The table makes the role of the residual visible. Its uniform decay, proved in Theorem 17.4, establishes the limit.

Problem 40 (A finite half-cell defect)**.** At $p = 5$, identify the sampled resets of B_4 on the half-cell 25-point grid and compute their combined point contribution. Evaluate $\mathcal{A}_5^{\mathrm{half}}$ by listing the nonzero incomplete blocks in its definition. Use the finite boundary identity to find $D_{25,1/2}[B_4]$. Compare the full answer with the point contribution.

18. Completed cycles and weighted interaction

A field built from many periodic families presents two different scales. An old family may have completed its source profile and repeat with zero mean, while a new family is still changing. Their interaction is controlled by the old primitive and the variation of its weighted partner.

This section constructs one such field and applies the weighted rules of Section 9. The single-family estimate retains the length of the old cycle. Summing it then determines how many families can be controlled together. That distinction between one interaction and the whole field is the central calculation.

18.1. **Tapered divisor families.** Each odd label r starts a family at length r and follows it through $2r, 4r, 8r$. The profile assigns the signed levels on those successive bands. The taper τ_S keeps the profile unchanged through S, reduces it linearly to zero at $2S$, and vanishes beyond that point. Multiplying the tapered profile's first difference at j by j produces the clock coefficients. This order matters at every band boundary and at $2S$.

Completed old profiles have zero period mean. Their changing weighted partners make this a test of the primitive estimate for a whole family of interactions.

The families are joined with the Möbius coefficients introduced in Section 14. Their signs remain in the combined field before it is squared. The same divisor-inversion identity will later connect the family source to its reset readings.

The family profiles below specify the complete source, including its taper and terminal coefficients. Their size, reset variation and screening bounds follow from these definitions. Let $t \geq 1$ be an integer, $S = 4t$, and

$$\tau_S(j) = \min(1, \max(0, (2S - j)/S)).$$

For each positive odd core r, write its profile as

$$\phi_r(j) = \frac{1}{r}\left(\mathbf{1}_{\{r \leq j < 2r\}} - \mathbf{1}_{\{2r \leq j < 4r\}} + \frac{1}{4}\mathbf{1}_{\{4r \leq j < 8r\}}\right).$$

Define the family source, raw field and joined source by

$$q_{t,r}(j) = j\Delta[\tau_S(j)\phi_r(j)], \qquad f_{t,r}(n) = \sum_j q_{t,r}(j)\{n/j\},$$

$$q_t(j) = \sum_{r \text{ odd}} \mu(r) q_{t,r}(j), \qquad p_t(n) = \sum_{r \text{ odd}} \mu(r) f_{t,r}(n), \qquad B_t = E_S[p_t].$$

Only $r < 2S$ occur in the parent, and every source includes its terminal coefficient at $j = 2S$. These are the prescribed Möbius families. Indeed, with $A(x) = \sum_{d \leq x} \mu(d)/d$,

$$A(j) - \frac{3}{2}A(j/2) + \frac{1}{2}A(j/4) = \sum_{r \text{ odd}} \mu(r)\phi_r(j).$$

This follows by separating odd d and $d = 2r$ in A; terms divisible by four have zero Möbius coefficient. Denoting the left side by $F(j)$ gives the original source $q_t(j) = j\Delta[\tau_S F](j)$. In particular $q_t(1) = 1$ is retained, and $\sum_j q_{t,r}(j)/j = \sum_j q_t(j)/j = 0$ by telescoping. The real-address fields are constant on open unit cells, so their continuous native integral in (18) uses exactly those clock readings.

At doubling put $\delta f_r = f_{2t,r} - f_{t,r}$ and $\delta p = p_{2t} - p_t$. Only $S/8 < r < 4S$ can change. If $8m \le S$, the complete old family is unchanged and equals

$$H_m(n) = \{n/m\} - 4\{n/(2m)\} + 5\{n/(4m)\} - 2\{n/(8m)\}.$$

Its eight-block profile is

$$H_m(n) = h(\lfloor n/m \rfloor \bmod 8), \qquad h = (0, -1, 2, 1, -1, -2, 1, 0).$$

Lemma 18.1 (Family size and changed reset variation)**.** *For every positive odd r, $|f_{t,r}(n)| \le 6$ at all integer addresses. The difference source $\delta q_r = q_{2t,r} - q_{t,r}$ satisfies*

$$\delta q_r(j) = 0 \ (j \le S \ \text{ or } j > 4S), \qquad \sum_j |\delta q_r(j)| \le 20, \qquad \sum_j \frac{\delta q_r(j)}{j} = 0. \tag{74}$$

Consequently $|\delta f_r| \le 12$ and $\Delta\delta f_r(n) = -\sum_{j|n} \delta q_r(j)$.

Proof. Put $r_j(n) = n \bmod j$ and $Z_j = r_j - r_{j+1}$. Finite summation by parts in the source, retaining the zero terminal profile, gives

$$f_{t,r} = V_{S,r} - 2V_{S,2r} + V_{S,4r}, \qquad V_{S,a} = \frac{1}{a} \sum_{j=a}^{\min(2a-1,2S-1)} \tau_S(j) Z_j.$$

For a nonempty sum ending at u,

$$aV_{S,a} = \tau_S(a) r_a + \sum_{j=a+1}^{u} (\tau_S(j) - \tau_S(j-1)) r_j - \tau_S(u) r_{u+1}.$$

After the first term the coefficients are nonpositive and their absolute sum is $\tau_S(a)$. The corresponding residues are less than $2a$, and $r_a < a$. Thus $-2\tau_S(a) \le V_{S,a} \le \tau_S(a)$, also for an empty sum. The signed combination $V_{S,r} - 2V_{S,2r} + V_{S,4r}$ lies between -6 and 6.

For the difference source put $\theta_S = \tau_{2S} - \tau_S$. It rises from zero at S to one at $2S$ and falls to zero at $4S$, with total discrete variation two. The product difference is exactly

$$\delta q_r(j) = \theta_S(j)\nu_r(j) + j\Delta\theta_S(j)\phi_r(j-1),$$

where $\nu_r = j\Delta\phi_r$ takes values $(1, -4, 5, -2)$ at $(r, 2r, 4r, 8r)$. Its absolute mass is twelve, and $j|\phi_r(j-1)| \le 4$. Therefore the difference mass is at most $12 + 4 \cdot 2 = 20$. The support and reciprocal identity follow from the product $\theta_S\phi_r$, including its terminal difference at $4S$. The field difference bound follows from the two family bounds; its reset identity follows from reciprocal neutrality. □

18.2. **Completed-cycle screening.** A completed old family repeats with zero mean. Against a constant partner, each complete period contributes zero. A changing weighted partner leaves work controlled by its variation, just as the alternating four-term example did in Section 7. Here the period has length $8m$, so the primitive bound must retain its dependence on m. That dependence determines how many families the estimate can control.

The completed profile has zero period mean and $|H_m| \le 2$. Its primitive $F_m(N) = \sum_{n=1}^{N} H_m(n)$ stays between $-m$ and $2m$. To see this, the values at block ends $N = jm-1$, $j = 1, \dots, 8$, are $0, -m, m, 2m, m, -m, 0, 0$. Between block ends the increments have constant sign, and complete periods sum to zero. Subtracting two primitive values consequently gives

$$\left|\sum_{n=a}^{b} H_m(n)\right| \le 3m \qquad (1 \le a \le b). \tag{75}$$

Theorem 18.2 (Completed-family screening)**.** *Let $S = 4t$, $t \ge 1$, let m be odd with $8m \le S$, and let r be any positive odd core. Then the complete interaction*

$$K_S(m, r) = \sum_{n>2S} w_n H_m(n)\delta f_r(n)$$

obeys

$$|K_S(m, r)| \le 129\frac{m}{S^2}. \tag{76}$$

Proof. Use Proposition 9.2 with $a = 2S + 1$, $u = H_m$ and $v = \delta f_r$. The primitive bound is $3m$ and $|v| \le 12$. By the complete signed reset identity,

$$\begin{aligned}\sum_{n\ge a} w_{n+1}|\Delta v(n+1)| &\le \sum_{j} |\delta q_r(j)| \sum_{\substack{k\ge a+1\\ j|k}} w_k \\ &\le 20\frac{2}{S^2}.\end{aligned}$$

For the last inequality, all changed clocks have $j > S$, and

$$\sum_{\ell\geq 1} w_{j\ell} \leq \frac{1}{j^2}\sum_{\ell\geq 1}\frac{1}{\ell^2} < \frac{2}{j^2} \leq \frac{2}{S^2}.$$

The reciprocal-square sum is less than two, since

$$\frac{1}{\ell^2} < \frac{1}{\ell-1} - \frac{1}{\ell} \qquad (\ell \geq 2).$$

The contribution from varying weights is $12w_a \leq 3/S^2$. Hence (22) gives

$$|K_S(m,r)| \leq 3m(3+40)/S^2 = 129m/S^2.$$

Its convergence hypotheses are verified by these same estimates, and the upper endpoint tends to zero. No period mean of the changed family has been subtracted. □

The bound uses a small primitive of the old field, not a universal sign of its interaction. The infinite tail and the actual jumps of its partner remain present throughout.

Problem 41 (A completed cycle under native weights)**.** Use the completed profile $H_1(n) = h(n \bmod 8)$ with $h = (0,-1,2,1,-1,-2,1,0)$. Compute its primitive at addresses $0,1,\ldots,8$. Find an interval attaining the bound $|\sum_{n=a}^{b} H_1(n)| \leq 3$. Compare its unweighted and native weighted sums over $1 \leq n \leq 8$. Explain which additional variation must be retained when a zero-mean cycle is observed with native weights.

18.3. **Summed screening bound.** For an integer $0 \leq M \leq S/8$, define the short old field and its work

$$p_{\leq M}(n) = \sum_{\substack{m\leq M\\ m \text{ odd}}} \mu(m)H_m(n), \qquad W_S(M) = 2\sum_{n>2S} w_n p_{\leq M}(n)\delta p(n).$$

The core sums are finite and the native products are absolutely summable, so

$$W_S(M) = 2\sum_{\substack{m\leq M\\ m \text{ odd}}}\ \sum_{\substack{S/8<r<4S\\ r \text{ odd}}} \mu(m)\mu(r)K_S(m,r).$$

There are at most $2S$ changed odd labels. Since the odd positive integers through M sum to $\lceil M/2\rceil^2$, the theorem gives

$$|W_S(M)| \leq \frac{516}{S}\lceil M/2\rceil^2 \leq 129\frac{(M+1)^2}{S}. \tag{77}$$

For

$$M_S = \min(\lfloor S/8 \rfloor, \lfloor \sqrt{S} \rfloor)$$

this is at most 516. It controls the interaction of all those short old cycles with every changed family. No cancellation among their Möbius signs is required for this portion.

The same proved estimate also allows a logarithmically longer range. With $\ell_t = 1 + \lfloor \log_2 t \rfloor$, take

$$M_t^\star = \min\bigl(\lfloor S/8 \rfloor, \lfloor \sqrt{S\ell_t^3} \rfloor\bigr).$$

Then (77) gives $|W_S(M_t^\star)| \le 516\ell_t^3$. This controls the complete work against every changed family through that cutoff. A range $S^{1/2+\epsilon}$ for fixed $\epsilon > 0$ would require a further saving.

Taking $M = \lfloor S/8 \rfloor$ in the same bound instead gives $O(S)$. The number and total cycle scale of the old families consume the saving in the pair estimate. This is an upper allowance from that estimate, not a lower bound on the actual signed work.

18.4. **Finite evaluation of an infinite interaction.** The screening bound concerns the complete series. A finite calculation can approximate it with an explicit error. For $N \ge 2S$ set

$$K_{S,N}(m,r) = \sum_{2S<n\le N} w_n H_m(n) \delta f_r(n).$$

The bounds $|H_m| \le 2$ and $|\delta f_r| \le 12$ give

$$|K_S(m,r) - K_{S,N}(m,r)| \le 24 \sum_{n>N} \frac{1}{n(n+1)} = \frac{24}{N+1}.$$

Thus the finite rational sum specifies an interval containing the infinite interaction. Taking $N \ge 2399$ makes its half-width at most $1/100$; a valid cutoff must also satisfy $N \ge 2S$. The calculation requires the actual fractional-clock readings at each integer, including resets. An equivalent evaluation uses the neutral floor sum from Section 9.

The finite interval and the all-scale theorem answer different questions. The interval encloses one chosen interaction; the screening theorem controls every admissible pair with its stated dependence on m, S.

18.5. **Large primitives and the screening bound.** Joining the old families before estimating gives a more informative version of the same method. Put

$$F_M(n) = \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) F_m(n), \qquad J_{S,M} = \sup_{n \ge 2S} |F_M(n) - F_M(2S)|.$$

The weighted variation of the complete changed field is at most $86/S$. Indeed there are at most $2S$ changed labels, and the proof of Theorem 18.2 bounds each weighted variation bracket by $43/S^2$. Therefore

$$|W_S(M)| \le 172 J_{S,M}/S. \tag{78}$$

The signs have now entered F_M. They do not, however, make this primitive uniformly small at every address.

Proposition 18.3 (Large primitives of the actual old source)**.** *For every integer $M \ge 10$ and every integer anchor $a \ge 0$,*

$$\sup_{n \ge a} |F_M(n) - F_M(a)| \ge \frac{1}{2} \sum_{\substack{M/2 < p \le M \\ p \text{ prime}}} (3p - 12).$$

Consequently this supremum is $\gg M^2/\log M$ as $M \to \infty$, uniformly in the anchor.

Proof. Let $\mathcal{P}$ be the primes in $(M/2, M]$. Each is at least seven and divides no odd label $m \le M$ other than itself. Let L be the least common multiple of all odd labels through M outside $\mathcal{P}$, with the empty least common multiple equal to one. Every $p \in \mathcal{P}$ is coprime to $8L$ and to the other selected primes.

Restrict addresses to $n \equiv 3 \pmod{8L}$. All unselected primitives then have fixed readings. For a selected prime, $F_p(4p-1) = 2p$. Put $d_p = 2$ if $p \equiv 1 \pmod 4$ and $d_p = 6$ if $p \equiv 3 \pmod 4$. Both $4p-1$ and $6p-1-d_p$ are congruent to three modulo eight. The second address lies in the block with increment -2, so $F_p(6p-1-d_p) = -p + 2d_p$.

The Chinese remainder theorem allows the residue modulo each selected p to be chosen independently while retaining $n \equiv 3 \pmod{8L}$. Choose one address with every selected primitive at its first reading, and another with every selected primitive at its second reading. Since $\mu(p) = -1$, the second joined reading minus the first is

$$\sum_{p \in \mathcal{P}} (3p - 2d_p) \ge \sum_{p \in \mathcal{P}} (3p - 12).$$

Both addresses can be moved beyond any anchor by adding their common period. At least one reading is at distance half this difference from $F_M(a)$. This proves the finite inequality. Finally the prime number theorem [7] gives $\sum_{M/2<p\le M} p \sim 3M^2/(8\log M)$, while the number of selected primes is $O(M/\log M)$. The asserted growth follows. □

For M of order $S^{1/2+\epsilon}$, any fixed $\epsilon > 0$, the uniform primitive estimate required to make (78) logarithmic is therefore false. The large readings may occur at very distant addresses. Their existence does not give a large native weighted interaction, because those addresses have small masses. A useful refinement must retain the weighted location of the primitive and its correlation with the changed source.

Problem 42 (The range of a summed screening estimate)**.** Starting with $|K_S(m,r)| \le 129m/S^2$, use at most $2S$ changed odd labels to recover an allowance for $W_S(M)$, with $M \le S/8$. At $S = 256$, evaluate that allowance for $M = 16$ and $M = 32$. For $M = \lfloor S^\alpha \rfloor$, $0 < \alpha < 1$, determine its order in S and the range of α for which it stays bounded. Identify the loss when every completed cycle is included. Does this calculation prove that the actual signed work grows at that order?

19. Native energy under refinement

At a doubled scale, both the clock field and its activation cutoff change. The observed energy loses the parent shell and gains the child field on the common tail. The field-change identity of Section 9 determines the whole increment before any families are separated.

Retain the fields of Section 18, with $S = 4t$ and $B_t = E_S[p_t]$. We first localize the full increment to a finite interval with a controlled tail. Aligning completed cycles then exposes the parent-shell credit, and combining resets gives a finite formula for the remaining work. Each step preserves the same energy change.

19.1. **Complete energy change.** Put $p_> = p_t - p_{\le M_S}$. The full moving-field identity is

$$B_{2t} - B_t = W_S(M_S) + R_S^{\mathrm{nat}}, \tag{79}$$
$$R_S^{\mathrm{nat}} = -\sum_{S<n\le 2S} w_n p_t(n)^2$$

$$+ \sum_{n>2S} w_n\big(2p_>(n)\delta p(n) + \delta p(n)^2\big).$$

The negative parent shell has not been split. The common tail includes all interactions with the remaining old cores and every interaction between changed cores. In particular $p_>$ contains intermediate completed cycles as well as families near the moving cutoff.

Write $\ell_t = 1 + \lfloor \log_2 t \rfloor$ for the number of doubling scales through t. It will measure the desired growth allowance.

19.2. **Finite localization of native work.** The weight of an address now supplies an effective limit on how far large primitive readings can matter. There are at most S odd labels in the parent and $2S$ in the child. Lemma 18.1 therefore gives, at every integer address,

$$|p_t(n)| \le 6S, \qquad |p_{2t}(n)| \le 12S.$$

These bounds are coarse in amplitude but useful in the distant tail.

Corollary 19.1 (Quadratic cutoff)**.** *Let $S = 4t$, $t \ge 1$, $Y = p_{2t} - p_t$, and $N \ge 2S$ be an integer. Define*

$$\begin{aligned} H_t(N) = &- \sum_{S<n\le 2S} w_n p_t(n)^2 \\ &+ \sum_{2S<n\le N} w_n\big(2p_t(n)Y(n) + Y(n)^2\big). \end{aligned} \tag{80}$$

Then

$$-\frac{36S^2}{N+1} \le B_{2t} - B_t - H_t(N) \le \frac{144S^2}{N+1}. \tag{81}$$

In particular, put $H_t = H_t(S^2)$. The assertions

$$B_{2t} - B_t \le C\ell_t^3 \quad (t \ge 1), \qquad H_t \le C'\ell_t^3 \quad (t \ge 1)$$

are equivalent with possibly different absolute constants.

Proof. Apply Theorem 9.4 with the two displayed field envelopes and native weights. Since $S \ge 4$, the cutoff S^2 is beyond $2S$. At that cutoff the omitted change lies strictly between -36 and 144. Thus an upper bound $C'\ell_t^3$ for H_t gives $(C' + 144)\ell_t^3$ for the complete increment. Conversely an upper bound $C\ell_t^3$ for the increment gives $(C + 36)\ell_t^3$ for H_t, because $\ell_t \ge 1$. □

The infinite tail has now been accounted for by an absolute allowance. The remaining arithmetic estimate concerns the finite addresses $S < n \le S^2$. A large primitive reading obtained by a distant congruence construction

need not occur inside this interval. Localization retains that distinction without requiring a uniform bound on the primitive over all addresses.

19.3. **Cycle alignment and parent credit.** The common start at $2S$ can cut an old cycle in the middle. Moving that family's start back to its previous period boundary makes its primitive start at zero. But the move restores readings between the two starts. Those fragments all lie in the parent shell already removed by the activation change. Their combined field Z_S must therefore be accounted for together with the old shell field. The construction below keeps that adjustment as a signed difference of two whole squares.

Here take $M = \lfloor S/8 \rfloor$, so every completed old family is included. The earlier cutoff M_S selected only the sufficiently short cycles controlled by screening. The alignment below is an identity for the larger range; estimating its signed work remains a separate task. Put $Y = \delta p$ and define

$$L_m = 8m \left\lfloor \frac{2S}{8m} \right\rfloor, \qquad Z_S(n) = \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) H_m(n) \mathbf{1}_{\{L_m < n \le 2S\}},$$

$$C_M = p_{\le M}, \qquad P_> = p_t - C_M,$$

$$\widetilde{W}_S = 2 \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) \sum_{n > L_m} w_n H_m(n) Y(n).$$

Since $8m \le S$, one has $S < L_m \le 2S$. Every restored fragment lies inside the same removed parent shell.

Proposition 19.2 (Cycle alignment and the parent credit)**.** *For the prescribed divisor source,*

$$B_{2t} - B_t = T_S + \widetilde{W}_S + \sum_{n > 2S} w_n \big(2P_>(n) Y(n) + Y(n)^2\big), \tag{82}$$

$$T_S = \sum_{S < n \le 2S} w_n \big(Z_S(n)^2 - (p_t(n) - Z_S(n))^2\big).$$

All series converge absolutely at each fixed scale. The sum of individual fragment energies satisfies

$$D_Z = \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m)^2 \sum_{L_m < n \le 2S} w_n H_m(n)^2 < \frac{1}{4}.$$

Proof. With $F(j) = A(j) - \frac{3}{2}A(j/2) + \frac{1}{2}A(j/4)$ as above, its untapered source is

$$\nu(j) = j\Delta F(j) = \mu(j) - 3\mathbf{1}_{2|j}\mu(j/2) + 2\mathbf{1}_{4|j}\mu(j/4).$$

The divisor sum of ν is $\mathbf{1}_{n=1} - 3\mathbf{1}_{n=2} + 2\mathbf{1}_{n=4}$. Reciprocal neutrality and the right-reading reset identity imply $p_t(1) = -1$, $p_t(2) = p_t(3) = 2$, and $p_t(n) = 0$ for $4 \le n \le S$. Applying the same argument to the child gives $p_{2t} = 0$ on $S < n \le 2S$, hence $Y = -p_t$ there.

Restoring each fragment changes the completed old work by

$$\widetilde{W}_S - W_S(M) = 2 \sum_{S<n\le 2S} w_n Z_S(n) Y(n) = -2 \sum_{S<n\le 2S} w_n Z_S(n) p_t(n).$$

Join this correction with the single negative parent-shell term in Proposition 9.3. Completing the square gives $-p_t^2 + 2Z_S p_t = Z_S^2 - (p_t - Z_S)^2$, proving (82). All fields are bounded at a fixed scale and the weights are summable.

Finally $|H_m| \le 2$ and there are at most M selected labels. Since

$$\sum_{S<n\le 2S} w_n < 1/(2S),$$

we have $D_Z < 2M/S \le 1/4$ when $M > 0$. For $M = 0$ the sum is zero. □

The fragment diagonal is paid uniformly. The joined fragment square also contains all interactions between distinct families, so D_Z cannot replace $\sum w_n Z_S(n)^2$. The useful quantity in the identity is the signed difference T_S itself.

For the aligned work the primitive F_m vanishes at L_m. The full infinite summation-by-parts identity is

$$\sum_{n>L_m} w_n H_m(n) Y(n) = \sum_{n>L_m} F_m(n)(w_n - w_{n+1}) Y(n) + \sum_j \delta q(j) \sum_{j\ell \ge L_m+2} w_{j\ell} F_m(j\ell - 1).$$

At a finite upper endpoint N, the first sum stops at $N-1$, the reset sum at $j\ell \le N$, and $F_m(N) w_N Y(N)$ must be added. The endpoint vanishes as $N \to \infty$. Weight variation is summable, and each of the finitely many reset progressions has summable weights. This justifies the displayed limit. Moving the initial boundary changes the reset cutoff as well as the primitive's initial value.

19.4. **Parent-shell response.** On the removed shell the actual divisor identity gives a shorter formula for the parent field. Retain the notation $F(n) = A(n) - \frac{3}{2}A(n/2) + \frac{1}{2}A(n/4)$.

Lemma 19.3 (Parent response before the second cutoff)**.** *For $S < n \leq 2S$, including the terminal address,*

$$p_t(n) = \frac{n(n-S)}{S}F(n) - \frac{1}{S}\sum_{k=S}^{n-1}(k-S)F(k). \tag{83}$$

Proof. Put $\nu(n) = n\Delta F(n)$. The divisor identity used in Proposition 19.2 gives $\sum_{j|n}\nu(j) = 0$ for $n > 4$. Every proper divisor of an address $S < n \leq 2S$ is at most S, where $q_t(j) = \nu(j)$. The actual right-hand reset therefore satisfies

$$\Delta p_t(n) = \nu(n) - q_t(n) = \frac{n-S}{S}\nu(n) + \frac{n}{S}F(n-1).$$

The product difference for $\tau_S F$ gives the second equality, including $n = 2S$. The right side of (83) is zero at $n = S$. Its first difference is the displayed expression, and $p_t(S) = 0$. Induction proves the formula. □

This is a consequence of the prescribed Möbius source. Replacing its family coefficients arbitrarily would remove the divisor identity on which the proof depends. The formula evaluates the negative shell in (80); that shell remains the square of the whole parent field.

The first-shell formula evaluates the parent at every admitted address, including those at which restored cycle fragments are active. In the cycle credit, the later retained-field square contributes to the same account as the early shell. Keeping both joined squares preserves the signed difference in (82).

19.5. **Joined resets on doubling blocks.** After localization, the remaining task is a finite sum of energy changes. Grouping addresses into doubling blocks makes it possible to compare the sizes of contributions at different distances. The boundary at the end of one block is the start of the next, so its endpoint term must cancel when the blocks are rejoined. Within a block we keep the combined reset amplitude and its interaction with the preceding field.

The finite interval can be partitioned at $2S, 4S, 8S, \ldots$, ending with S^2. To evaluate its work by resets, combine the clocks that reset at the same

address. Define

$$a(n) = \sum_{\substack{j|n \\ j\le 2S}} q_t(j), \qquad d(n) = \sum_{\substack{j|n \\ j\le 4S}} \delta q(j), \qquad g(n) = p_{2t}(n)^2 - p_t(n)^2.$$

The source has been extended by zero outside its support. Reciprocal neutrality gives $\Delta p_t(n) = -a(n)$ and $\Delta Y(n) = -d(n)$.

Proposition 19.4 (Finite signed reset work)**.** *For the actual source, put* $J(n) = g(n) - g(n-1)$. *Then*

$$\begin{aligned} J(n) = & -2Y(n-1)a(n) - 2\big(p_t(n-1) + Y(n-1)\big)d(n) \\ & + 2a(n)d(n) + d(n)^2. \end{aligned} \tag{84}$$

For any integers $2S \le u < v$,

$$\sum_{u<n\le v} w_n g(n) = \frac{g(u)}{u+1} - \frac{g(v)}{v+1} + \sum_{n=u+1}^{v} \frac{J(n)}{n}. \tag{85}$$

Consequently the complete finite head is

$$H_t = -\sum_{S<n\le 2S} w_n p_t(n)^2 - \frac{p_t(2S)^2}{2S+1} - \frac{g(S^2)}{S^2+1} + \sum_{n=2S+1}^{S^2} \frac{J(n)}{n}. \tag{86}$$

Proof. At a reset, the old field changes from $p_t(n-1)$ to $p_t(n-1)-a(n)$, and the child changes from $p_t(n-1)+Y(n-1)$ to that value minus $a(n)+d(n)$. Subtract the two square changes. The old $a(n)^2$ cancels, leaving (84). Its mixed terms retain both the inherited field and the interaction between the two combined amplitudes.

For the weighted identity write $g(n) = g(u) + \sum_{k=u+1}^{n} J(k)$ and interchange the finite sums. The inner weight sum is $\sum_{n=k}^{v} w_n = 1/k - 1/(v+1)$. Collecting the coefficients of $g(u)$ and $\sum J = g(v) - g(u)$ gives (85).

Apply this identity on the consecutive doubling blocks. Each interior term $-g(v)/(v+1)$ cancels the next block's $g(u)/(u+1)$. The remaining initial value is $g(2S) = -p_t(2S)^2$, since the child vanishes there. Add the single parent shell to obtain (86). □

No intermediate block endpoint has acquired a separate charge. Formula (84) also retains interactions between distinct divisors at the same address. Splitting a and d into individual clock magnitudes would lose those signs.

The immediate envelope estimate on a block is

$$\sum_{u<n\le v} w_n g(n) \le 144S^2\left(\frac{1}{u+1}-\frac{1}{v+1}\right).$$

Its sum from $2S$ to S^2 is of order S. Thus localization and regrouping have identified a finite signed target, but the desired logarithmic estimate still requires a bound for the actual correlations in (84). The parent shell and all block signs remain available together in (86).

19.6. **Uniform increment estimate.** With $\ell_t = 1 + \lfloor \log_2 t \rfloor$, the remaining estimate is

$$\boxed{R_{4t}^{\mathrm{nat}} \le C\ell_t^3 \qquad (t \ge 1).} \tag{87}$$

Since $|W_S(M_S)| \le 516$, this estimate is equivalent, up to a changed absolute constant, to $B_{2t} - B_t \le C'\ell_t^3$. The estimate is open. Summing it along a doubling chain would give a fourth-power logarithmic energy bound; that energy bound alone would not control each increment.

Uniform control must account for the changing field. Fixed-field quadrature constants depend on its boundary data and curvature norms. The half-cell parity transfer addresses another field and observation; it does not bound the native remainder in (79).

Problem 43 (Two-clock refinement and a complete tail)**.** Use the two-clock model $p(n) = \{n\} - 2\{n/2\}$ at integer addresses, with its actual right-hand readings. Replace it by $q(n) = p(n) + 1$ and move the native cutoff from two to four. Compute $E_4[q] - E_2[p]$ by separating the removed parent shell, inherited interaction and new square. Evaluate the infinite tails. Then form the finite head through $n = 10$ and use $|p|, |q| \le 1$ to enclose the omitted change. Decide its sign from that finite enclosure alone.

20. Invariant readings on the parent shell

The parent-shell credit contains a negative retained-field square. A lower bound for that square can therefore give an upper bound for the energy change. This applies the invariant-reading method of Section 8 to a concrete family of allowed additions.

The later families vanish on the inherited prefix. On the parent shell, their first nonzero readings form a triangular matrix. That structure lets us construct local invariants and compare the information they retain with

the full constrained minimum. A finite counterexample will show exactly what a sparse collection of readings can miss.

20.1. **Four-reading invariants.** The retained field in the parent credit is $p_t - Z_S$. The inherited prefix fixes the coefficients of the old families. Indeed, at an odd address $m \le S$ the untapered reset amplitude is $\sum_{r|m} c_r$. The prescribed prefix makes it one at $m = 1$ and zero at all other odd $m \le S$; divisor inversion gives $c_r = \mu(r)$ for odd $r \le S$. Every odd family with $S < r < 2S$ vanishes through address S. These later families have a particularly simple response on the shell $I_S = \{S+1, \dots, 2S\}$. The following construction develops the invariant readings fixed by that prefix.

Lemma 20.1 (Later family on the parent shell)**.** *For odd $S < r < 2S$ and integer $1 \le n \le 2S$,*

$$f_{t,r}(n) = \begin{cases} 0, & n < r, \\ \dfrac{n(n+1) + r(r-1) - 4Sr}{2Sr}, & r \le n \le 2S. \end{cases}$$

The $S/2$ later families are linearly independent as vectors on I_S.

Proof. Reciprocal neutrality gives $f_{t,r}(n) = -\sum_j q_{t,r}(j)\lfloor n/j \rfloor$. For $r > S$ the source is supported on $r \le j \le 2S$, and $n \le 2S < 2r$. Each nonzero floor is therefore one. The field is zero before r, and otherwise it is

$$-\sum_{j=r}^{n} j\Delta\left[\frac{2S-j}{Sr}\right],$$

where the profile is zero before r and its terminal value at $2S$ is zero. Summation by parts gives the displayed formula. In particular $f_{t,r}(r) = -(2S-r)/S \neq 0$. At the ordered entrance addresses the evaluation matrix is triangular with nonzero diagonal, proving independence. □

Write the inherited retained field and an arbitrary later addition as

$$v_S = \sum_{\substack{r \le S \\ r \text{ odd}}} \mu(r) f_{t,r} - Z_S, \qquad Ux = \sum_{\substack{S<r<2S \\ r \text{ odd}}} x_r f_{t,r}.$$

The actual retained field is obtained by taking $x_r = \mu(r)$. Use the native shell norm $\|v\|_w^2 = \sum_{n \in I_S} w_n |v(n)|^2$.

The construction now seeks a combination of four readings that every later addition leaves unchanged. On a four-address block beginning at an odd address, families already active lie in the span of the two profiles 1

and $n(n+1)$. Only one additional odd family can enter strictly inside the block. Annihilating those three contributions gives three homogeneous linear conditions on four coefficients. The following choice solves them; the proof checks that the resulting reading is nonzero.

Take four consecutive addresses beginning at an odd $q > S$, with $q+3 \leq 2S$. Put $G(n) = n(n+1)$ and $s = q+2$. Define

$$a_2 = f_{t,s}(s+1), \qquad a_3 = -f_{t,s}(s),$$
$$a_1 = -\frac{[G(q+2)-G(q)]a_2 + [G(q+3)-G(q)]a_3}{G(q+1)-G(q)},$$
$$a_0 = -a_1 - a_2 - a_3.$$

The reading on this block is $L_q(v) = \sum_{i=0}^{3} a_i v(q+i)$.

Proposition 20.2 (Inherited four-reading bound)**.** *Each L_q is nonzero and satisfies $L_q(Ux) = 0$ for every later coefficient vector x. For the disjoint blocks $q = S+1, S+5, \ldots, 2S-3$, put*

$$\mathcal{L}_S = \sum_q \frac{|L_q(v_S)|^2}{\sum_{i=0}^{3} a_i^2/w_{q+i}}. \tag{88}$$

Then, at every scale and for every x,

$$\|v_S + Ux\|_w^2 \geq \mathcal{L}_S. \tag{89}$$

Proof. The coefficients satisfy $\sum_i a_i = 0$ and $\sum_i a_i G(q+i) = 0$. A later family entering at or before q is a linear combination of these two modes throughout the block. The only later entrance strictly inside the block is $s = q+2$; the definitions of a_2, a_3 give $a_2 f_{t,s}(s) + a_3 f_{t,s}(s+1) = 0$. Families entering after the block vanish on it. This proves the annihilation assertion for every later family, hence for their whole span. Moreover $a_3 = (2S-s)/S > 0$, so the functional is nonzero. Theorem 8.3 applied to the disjoint blocks gives (89). □

For example, at $S = 4$ the only block is $\{5,6,7,8\}$. Multiplying its coefficients by 84 gives the simpler reading

$$L(v) = 56v(5) - 80v(6) + 3v(7) + 21v(8).$$

It annihilates both later families, of cores five and seven. The reciprocal-weight denominator is

$$56^2(5\cdot 6) + 80^2(6\cdot 7) + 3^2(7\cdot 8) + 21^2(8\cdot 9) = 395136.$$

Thus any inherited field v on this block obeys $\|v + Ux\|_w^2 \geq |L(v)|^2/395136$.

20.2. **Local bounds and the constrained minimum.** An invariant reading measures some of the energy that later additions cannot remove. A collection of readings can miss part of that energy. The comparison below tests whether the particular four-reading collection captures enough to pay for the restored fragments. Its failure will therefore test the chosen collection, not erase the retained energy.

The four-reading theorem gives an energy lower bound. To bound the parent adjustment by its fragment diagonal, a sufficient comparison would be

$$\|Z_S\|_w^2 - D_Z \leq \mathcal{L}_S. \tag{90}$$

Together with (89), it would imply $T_S \leq D_Z < 1/4$. This particular comparison fails for the actual source.

At $t = 51$, hence $S = 204$, exact rational evaluation gives

$$\|Z_S\|_w^2 - D_Z > \frac{3}{1000}, \qquad \mathcal{L}_S < \frac{1}{1000}.$$

At the same scale,

$$\|Z_S\|_w^2 < \frac{1}{100}, \qquad \|p_t - Z_S\|_w^2 > \frac{7}{100},$$

so the full parent adjustment is negative. The local readings detect only part of the retained energy. The same loss occurs at the scales shown below; decimals are rounded displays of exact rational values.

t	Fragment excess	$\mathcal{L}_S$	$\mathcal{L}_S - (\|Z_S\|_w^2 - D_Z)$
51	0.003253615	0.000729123	−0.002524492
56	0.005590292	0.000506243	−0.005084050
61	0.004029631	0.000627639	−0.003401992
62	0.004166453	0.000596618	−0.003569834

There is a dimensional reason to distinguish this local budget from the whole retained energy. The shell has S readings and the later span has dimension $S/2$. Its full annihilator therefore has dimension $S/2$, whereas the disjoint four-reading construction uses only $S/4$ independent functionals. It samples half of those directions; there is no corresponding assertion that it captures half the energy.

If a matrix A spans the full annihilator, Proposition 8.4 gives the minimum retained energy as

$$(Av_S)^*(AW^{-1}A^*)^{-1}(Av_S).$$

This uses all invariant readings. The uniform comparison of the fragment excess with this minimum remains open. The failed local comparison does not settle it. The full increment also retains the aligned work and the remaining joined field change in (82).

Problem 44 (A complete set of shell invariants)**.** At $S = 4$ use the later-family space $V = \operatorname{span}\{f_{1,5}, f_{1,7}\}$ on addresses $5, 6, 7, 8$ and the native shell weights. For the test inherited vector $v = (10, 7, 0, 0)^T$, evaluate the four-reading functional $L = (56, -80, 3, 21)$. Construct a second independent functional J that annihilates V, using only the first two addresses. Form the joint matrix for rows L, J and determine the minimum of $\|v + h\|_w^2$ over $h \in V$. Explain the information missed by L alone.

21. Boundary and energy accounting

A boundary-energy problem specifies actual readings and their observation masses. Unity supplies a fixed target through subdivision and approximation. Boundary values, combined jumps and inherited interactions determine the energy comparison.

The examples show the consequences of these choices. A changed reading reverses the sign of a defect; two individually exact pulse observations acquire a defect through their mixed term. Invariant readings identify energy that the permitted additions cannot remove. In the moving-window example, a primitive bound and envelope variation control the complete growing family.

For a new application, the same decisions remain available. Local boundary data support a periodic decomposition. Running sums support weighted estimates. Congruence intersections and primitive shells make arithmetic structure explicit. The useful representation is the one whose remaining terms can be evaluated or bounded with the data of the problem.

At growing scales, parity transfer establishes the quarter-limit for the shifted carry field. Completed-cycle screening bounds a range of native interactions. Localization and combined reset work retain the parent-shell credit in the remaining energy change, whose uniform estimate (87) remains open.

Hints and solutions

The answers follow the problem numbers, grouped by part. Each answer heading links back to its question. Where a hint is provided, try it before reading the full solution.

Part I. Boundary readings and sampled energy

Problem 1. *Solution.* Only the first squared reading changes. The sampled energy is $c^2/3+2/27$, and the discrepancy is $7/27-c^2/3$. It vanishes exactly when $c=\pm\sqrt{7}/3$. No integral needs to be recomputed.

Problem 2. *Hint.* Use the observation masses both in the mean and in the squared error. Subtract unity before squaring the error.

Solution. The weighted mean is $1+(a+2b)/3$, so it equals one when $b=-a/2$. In that case

$$\frac{1}{3}(1+a)^2+\frac{2}{3}(1-a/2)^2=1+\frac{a^2}{2}, \qquad \frac{1}{3}a^2+\frac{2}{3}(-a/2)^2=\frac{a^2}{2}.$$

The weighted detail has mean zero, so its mixed term with unity vanishes. For $a=2$, the readings are $3,0$, the energy is three and the error energy is two. With $b=-a$, the mean is $1-a/3$, the energy is $1-2a/3+a^2$, and the error energy is a^2. Opposite changes need not have zero weighted mean. The total mass stays one in both cases. Copying the parent reading one would also preserve its energy; changed child readings require the new square.

Problem 3. *Solution.* Equation (1) gives $D_q[f_q]=q^{2\alpha-1}(1-2/(3q))$. The three cases are $\alpha<1/2$, $\alpha=1/2$ and $\alpha>1/2$, respectively.

Problem 4. *Hint.* Determine whether each grid samples zero.

Solution. Check whether the exceptional point is sampled. The continuous energy is zero. The unshifted grid has sampled energy $4/4=1$ and defect -1. The half-cell grid misses zero, so both its sampled energy and defect vanish. The assigned value at an atom distinguishes the observations.

Problem 5. *Hint.* Expand the new squared readings and keep the change of observation separate.

Solution. Keep the masses separate from the field values. The old energy is one; the new field is $(1,0,1)$ and its energy is $2/3$. The observation change is $3-1=2$, the inherited interaction is $-10/3$, and the new square is one. Their sum is $-1/3$, agreeing with the direct subtraction.

Problem 6. *Hint.* Form the squared readings before taking their average. At a boundary, square each trace separately.

Solution. At 0 the traces of f are $3, 1$; at $1/2$ they are $1, 3$. The actual reading is two at both points. Thus the corresponding traces of h are $9, 1$ and $1, 9$, and its actual reading is four. Consequently $a_0(h) = a_{1/2}(h) = 1$ and $[h]_0 = -8$, $[h]_{1/2} = 8$. All slope traces vanish. The continuous energy is five. On the three-point grid the readings are $(2, 1, 3)$; on the four-point grid they are $(2, 1, 2, 3)$. Hence

q	$Q_q(f^2)$	$D_q[f]$	$P_q[f]$
3	14/3	1/3	1/3
4	9/2	1/2	1/2

Only zero is a sampled boundary for $q = 3$; both are sampled for $q = 4$. In both cases $Q_q f = 2$, so $(Q_q f)^2 = 4$ differs from the observed energy. Here the remaining signed defect is zero.

Problem 7. *Hint.* Apply boundary decomposition to the squared field.

Solution. Apply the boundary formula to $h = x^2$, rather than to f. The point correction is $1/2$, the value jump is -1, the derivative jump is -2, and the regular curvature is the constant two. The value-jump term vanishes on the unshifted grid. The regular term has zero Bernoulli mean. Thus

$$D_q[f] = \frac{1}{2q} - \frac{1}{6q^2}.$$

Directly, subtract the sampled sum from the integral:

$$D_q[f] = \frac{1}{3} - \frac{1}{q^3} \sum_{a=0}^{q-1} a^2 = \frac{1}{2q} - \frac{1}{6q^2}.$$

Problem 8. *Hint.* Use $\sin^2 t = (1 - \cos 2t)/2$ and the finite Fourier-mode average.

Solution. The integral is $1/2$. Every reading at $q = 2m$ is zero, so the defect is $1/2$. At $q = 2m + 1$, the mode of frequency $2m$ has zero grid average; the sampled energy is $1/2$ and the defect is zero. The same argument gives zero defect for every fixed-field grid $q > 2m$. There are no jumps, exceptional readings or derivative jumps. The regular-curvature integral in Theorem 4.1 therefore supplies the whole defect on the resonant grid. Along $q = 2m$ the frequency grows with the grid, so the fixed-field conclusion does not apply to that sequence.

Problem 9. *Solution.* Point-value replacement and the single-clock formula give

$$D_q[g] = D_q[f] - \frac{1}{q} = -\frac{2}{3q^2}.$$

The two energy traces at zero equal one, as does the new sampled energy. The point correction has disappeared. The remaining derivative term is negative, so the grid records more energy than the integral.

Problem 10. *Hint.* Write F on the two open half-intervals and combine the coefficients at a shared reset before squaring.

Solution. The possible resets are $0, 1/2$. Their combined coefficients are $C_0 = 1 - 1 = 0$ and $C_{1/2} = -1$. The field is

$$F(x) = \begin{cases} -2x, & 0 < x < 1/2, \\ 2 - 2x, & 1/2 < x < 1, \end{cases} \qquad F(0) = F(1/2) = 0.$$

It has continuous energy $1/3$. Its four readings are $(0, -1/2, 0, 1/2)$, so the sampled energy is $1/8$ and the defect is $5/24$. The first component has the same four readings and continuous energy; the second has continuous energy $1/3$ and all four readings zero. Thus

$$D_4[f] + D_4[g] = \frac{13}{24}, \qquad 2\operatorname{Re} K_4(f, g) = -\frac{1}{3}.$$

At zero the resetting coefficients cancel, so that point has no combined jump or point-energy correction. Squaring the separate coefficients would lose this cancellation.

Problem 11. *Solution.* Each interval has length $1/2$ and contains two of the five sample points. Each individual defect is $1/2 - 2/5 = 1/10$. The intersection has length $1/4$ and contains one sample, so its defect is $1/20$. Since $f^2 = \mathbf{1}_I + \mathbf{1}_J + 2\mathbf{1}_{I\cap J}$,

$$D_5[f] = \frac{1}{10} + \frac{1}{10} + 2\frac{1}{20} = \frac{3}{10}.$$

Directly, the field has readings $0, 0, 1, 2, 1$, giving sampled energy $6/5$ and continuous energy $3/2$. Adding only the two individual defects would miss $1/10$ of the discrepancy.

Problem 12. *Hint.* Inspect the square of the combined field before replacing either reading.

Solution. Square the combined field first. Its square is one everywhere, so $D_q[f] = 0$. Both changed points are sampled when q is even. Removing their two unit squares reduces the sampled energy by $2/q$ and leaves the integral unchanged. Hence $D_q[g] = 2/q$.

Problem 13. *Hint.* Count the grid points in the half-open pulse separately for each residue class of q.

Solution. Count samples before scaling. The number is $\lceil q/3 \rceil$. For $q = 3m$ the defect is zero; for $q = 3m+1$ it is $-2/(3q)$. Both fixed-field defects tend to zero. Multiplication of the field by $\sqrt{q}$ multiplies its defect by q, giving the distinct values zero and $-2/3$. The growing family has no common limit along these sequences.

Part II. Field and observation changes

Problem 14. *Solution.* Extend f through $b+1$. The observation change contributes

$$-v_{a+1} f(a+1)^2 + v_{b+1} f(b+1)^2.$$

On the new window add

$$\sum_{a+1<n\le b+1} v_n(2f(n)g(n) + g(n)^2).$$

These are exactly the three terms of (3). Changing masses at retained addresses would add their differences times $f(n)^2$ to the first term.

Problem 15. *Hint.* Apply summation by parts to the product $w_n v(n)$, not to $v(n)$ alone.

Solution. The primitive is $(1, 0, 1, 0, 1)$ and the direct sum is

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20} + \frac{1}{30} = \frac{2}{5}.$$

Summation by parts gives

$$T(5)w_5 + \sum_{n=1}^{4} T(n)(w_n - w_{n+1}) = \frac{1}{30} + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{12} - \frac{1}{20}\right) = \frac{2}{5}.$$

The partner variation vanishes, but the weight variation does not. Dropping the endpoint gives $11/30$; dropping the interior terms gives $1/30$. Neither is the observed sum.

Problem 16. *Hint.* Track the offset of the three distinct arguments after dilation, as well as their multiplicity.

Solution. For $r = 2$, the half-cell grid visits $1/6, 1/2, 5/6$, each twice. Their readings are $-2/3, 0, 2/3$, so the sampled energy is $8/27$ and the defect is $1/27$. For $r = 4$, it visits $0, 1/3, 2/3$, each twice. Their readings are $0, -1/3, 1/3$, so the sampled energy is $2/27$ and the defect is $7/27$. Dilation preserves the continuous energy $1/3$. At offset zero both dilations visit $0, 1/3, 2/3$ twice and give defect $7/27$. The reduced grid has three points in both cases, but its offset is $\{r\theta/2\}$: it is $1/2$ for $r = 2$ and zero for $r = 4$ when $\theta = 1/2$.

Problem 17. *Solution.* Every entry of the defect matrix is $d_q = 1/q - 2/(3q^2)$, so

$$H = d_q \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad U^* H U = \begin{pmatrix} 2d_q & 0 \\ 0 & 0 \end{pmatrix}.$$

The symmetric coordinate represents $\sqrt{2} f_1$; the antisymmetric coordinate represents the zero field. The sampled point matrix is $q^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The same basis change diagonalizes it. Its trace is preserved, but its two diagonal energies are unequal. This elementary case distinguishes a unitary invariant from an equal-distribution assertion.

Problem 18. *Solution.* When $q = m$, every sampled point is a reset. The sample energy is zero and $D_m[f_m] = 1/3$. When $q = m+1$, the two integers are coprime. Dilation reduces the samples to those of the unit clock, giving

$$D_{m+1}[f_m] = \frac{1}{m+1} - \frac{2}{3(m+1)^2} \longrightarrow 0.$$

The number of samples alone does not identify the limiting regime. Its common divisors with the growing clock frequency matter.

Problem 19. *Solution.* Here $b = (-2, 1)^T$ and

$$G = \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix}, \qquad G^{-1} = \frac{1}{20} \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}.$$

The joint bound is $b^* G^{-1} b = 7/10$. The kernel of A consists exactly of affine sequences, so this is the minimum. It is attained at $z = (-4, 7, -2, -1)/10$, obtained by adding $h(i) = (i-5)/10$. The two separate bounds sum to $4/6 + 1/6 = 5/6$, which exceeds the attained energy. Their common addresses require the mixed entries of G.

Problem 20. *Hint.* Find the weighted mean, then test two differences that annihilate constants.

Solution. A constant adjustment removes the weighted mean, which is $1/4$. The minimizing addition is $-1/4$, and its energy is

$$(3/4)^2 + 2(-1/4)^2 + (-1/4)^2 = 3/4.$$

Use readings $v_1 - v_2$ and $v_2 - v_3$. With $W = \operatorname{diag}(1, 2, 1)$,

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$

Since $Af = (1, 0)^T$ and $\ker A$ is the constant space, the joint bound is the attained minimum $3/4$.

Problem 21. *Hint.* Minimize the energy over a constant addition and compare it with the proposed reading.

Solution. The single reading gives the valid lower bound zero. A real constant addition c has energy $c^2 + (1 + c)^2 + c^2 = 3(c + 1/3)^2 + 2/3$. An imaginary part adds three times its square, so the minimum over complex additions is also $2/3$. The first and third readings agree, but their common difference from the middle reading survives every constant addition. That is the information the chosen functional misses.

Problem 22. *Hint.* A constant reading has native tail mass $1/(S + 1)$.

Solution. Use $\sum_{n>S} w_n = 1/(S + 1)$. The old energy is $1/2$ and the new energy is $(1/2)^2/4 = 1/16$, so the change is $-7/16$. Its three contributions are $-1/4$, $-1/4$ and $1/16$. They come respectively from the removed addresses two and three, the constant mixed term on the remaining tail, and the new square.

Problem 23. *Hint.* Bound the omitted new and old energies separately before subtracting.

Solution. The omitted change lies in $[-4/(N + 1), 9/(N + 1)]$. Both endpoints have absolute value at most $1/10$ when $N + 1 \geq 90$; the least admissible cutoff from these envelopes is $N = 89$. A head of $-1/10$ then gives a nonpositive complete change, with interval $[-13/90, 0]$. Strict negativity does not follow. With envelopes $2S, 3S$ the condition is $N + 1 \geq 90S^2$, so $N = 90S^2 - 1$ suffices and is at least $2S$.

Problem 24. *Solution.* In the notation of Section 10, the exact expression is $U_N + 2N^\alpha V_N + N^{2\alpha} U_N$. The proof of Theorem 10.1 gives $U_N = 1/(2N) + O(N^{-2})$ and $V_N = O(N^{-2})$. For $\alpha < 1/2$ all terms tend to zero. At $\alpha = 1/2$ the limit is $1/2$. For $\alpha > 1/2$ the last term is asymptotic to $\frac{1}{2}N^{2\alpha-1}$ and dominates the other two, so the energy diverges. The threshold comes from the observation mass as well as the field's amplitude.

Problem 25. *Hint.* Inspect partial sums of both the carrier and its squared fluctuation over one period.

Solution. Inspect the range of each period's partial sums. For u the range has width one. For $u^2 - 2/3$, with period $(1/3, 1/3, -2/3)$, it has width $2/3$. Thus $P = 1$, $R = 2/3$, $c = 2/3$ are admissible. All three products h_i equal one, so their variations vanish and $\|h_i\|_{\mathrm{BV},*} = 1$. The theorem gives

$$\left| E_N - \frac{2}{3} - \frac{1}{N} \right| \le \frac{2}{3N} + \frac{2}{N^{3/2}}.$$

The limit is $2/3$. At $N = 4$ the readings are $(-1, 1, 3, -1)$, with common weight $1/16$, giving $E_4 = 3/4$.

Problem 26. *Hint.* Identify whether the changed object is a point reading, weighted partner, tail or growing field.

Solution. Match the operation to the changed data. An isolated reading uses point-value replacement and its sample mass. Periodic work uses summation by parts with an anchored carrier primitive and the variation of its weighted partner. A finite head uses localization with both tail energies bounded. For a growing family, track every boundary amplitude, curvature norm and other field-dependent constant as a function of scale. Uniform control of those constants is the additional requirement.

Problem 27. *Hint.* Retain the mixed term before substituting the two amplitudes.

Solution. Retain the mixed energy and put $M = q(a+b)^2/2$. The continuous energy is $M - ab$. The sampled energies are $M - 2ab$ for $\theta < 1/2$ and M for $\theta \ge 1/2$, so the defects are $ab, -ab$. They vanish for every offset exactly when $ab = 0$. The difference gives defects $-1, 1$; the sum gives $1, -1$ and continuous energy $2q - 1$.

Problem 28. *Hint.* Telescope the weights for the two squares. Pair successive terms of the alternating tail before estimating the mixed term.

Solution. Let $T_N = \sum_{n>N} w_n(-1)^n$. Expansion of the whole square gives

$$\mathcal{E}_N = \frac{a^2+N}{N+1} + 2a\sqrt{N}\,T_N.$$

The weights decrease to zero. Pairing adjacent terms, starting with the first term or with the second, places the alternating sum between zero and its first signed term. Hence

$$|T_N| \le w_{N+1} = \frac{1}{(N+1)(N+2)},$$

$$\left|\mathcal{E}_N - 1 - \frac{a^2-1}{N+1}\right| \le \frac{2|a|\sqrt{N}}{(N+1)(N+2)}.$$

All series converge absolutely for each fixed N, and the displayed allowance proves $\mathcal{E}_N \to 1$. For $a = 1$, the envelope gives

$$\sum_{n>M} w_n |p_N(n)|^2 \le \frac{(1+\sqrt{N})^2}{M+1} \le \frac{4N}{M+1}.$$

The integer choice $M_N = 4N^2$ therefore suffices. A cutoff proportional to N leaves only an $O(1)$ allowance from this envelope. The desired $1/N$ accuracy requires a quadratic cutoff by this particular method.

Part III. Finite arithmetic tools

Problem 29. *Solution.* Both components are read only at $r = 25$. Their combined reading is $2 - 3 = -1$, so the energy is one. The ordered terms are $4, 9, -6, -6$, whose sum is one. Discarding the two mixed signs would give 25.

Problem 30. *Hint.* Solve the new pair of congruences and then test the observation interval.

Solution. The new residue is 39 modulo 56, since $39 \equiv 4 \pmod 7$ and $39 \equiv -1 \pmod 8$. Both systems are compatible, but the original kernel on $24 < r < 27$ is one and the new kernel is zero. Changing an orientation can change the observed intersection.

Problem 31. *Hint.* List the common rows first, and then count their observation masses.

Solution. The common class is 10 modulo 12, so both rows ten and twenty-two are observed. Their total mass is three. The first component occupies nine rows and the second six; the extra mass at row ten gives

$$K_v = \begin{pmatrix} 10 & 3 \\ 3 & 7 \end{pmatrix}.$$

The two energies are

$$(2,-1)K_v(2,-1)^T = 40-12+7 = 35, \qquad (2,1)K_v(2,1)^T = 40+12+7 = 59.$$

The sign change adds twenty-four through the mixed terms. The observation interval is longer than the common period twelve, so the intersection kernel must sum both common rows and their actual masses.

Problem 32. *Hint.* Balance the two powers in the allowance.

Solution. Write $J = p^\theta$. The costs have exponents θ and $1-\theta/3$, which agree at $\theta = 3/4$. Taking an adjacent integer width gives allowance $O(p^{3/4})$ and normalized rate $O(p^{-1/4})$. This conclusion uses the two assumed costs; it does not estimate a new smooth family.

Problem 33. *Hint.* The arguments may differ by an integer. Use $|e(t)-1| = 2|\sin(\pi t)|$ for the omitted factor.

Solution. Both inverses are three in their respective moduli. The left argument is $9/7$ and the full right argument is $-9/5+3/35 = -12/7$; their difference is three. Thus the two phases agree. Omitting the correction leaves absolute error

$$|e(9/7) - e(-9/5)| = |e(3/35) - 1| = 2\sin(3\pi/35).$$

In general the other phase has unit magnitude, so the error is

$$2|\sin(\pi A/(rc))| \le \frac{2\pi|A|}{rc}.$$

It is also at most two. This estimate concerns the phase correction alone; multiplying or summing it requires retaining the accompanying amplitudes.

Problem 34. *Solution.* The coefficients are $1,2,4,4$. For $m > 1$ there are two orientations of each coprime pair (k,m), $1 \le k < m$. The multiplicities at cutoff four are $4,2,1,1$. Thus $\mathcal{A}(4) = 4+4+4+4 = 16$, as direct counting gives.

Problem 35. *Hint.* Compute the four primitive shells and retain the prime-power contribution at four.

Solution. The nonzero direct terms are $4\log 2 + \log 3$. The shell formula gives $\Psi(4)+2\Psi(2)$ because the later arguments are below two. Since $\Psi(4) = 2\log 2+\log 3$, the same value follows. Both calculations retain the repeated contribution of the prime two at the prime power four.

Problem 36. *Hint.* Reduce each nonzero pair by its common divisor. Only three primitive pairs remain.

Solution. The nonzero primitive pairs are $(1,1),(1,2),(2,1)$. Thus $P(1) = 2$, $P(2) = -2$ and $P(m) = 0$ for $m > 2$. Common dilation preserves each of the three conditions defining R. The prefix is

$$\mathcal{A}(M) = 2M - 2\lfloor M/2\rfloor = 2\lceil M/2\rceil,$$

so its first six values are $2, 2, 4, 4, 6, 6$. The nonzero logarithmic terms at $N = 6$ give

$$\mathcal{K}(6) = 4\log 2 + 2\log 3 + 2\log 5 - 2\log 6 = 2\log 10.$$

The shell formula gives the same value as $2\Psi(6)-2\Psi(3)$, since $\Psi(6)-\Psi(3) = \log 2 + \log 5$. At every even entrance M, the added diagonal contribution two is canceled by the two new off-diagonal readings $-1, -1$.

Problem 37. *Hint.* Normalize the later term by the proposed allowance at n.

Solution. Solving for the value at n divides by c. The later term has argument an/b, so its absolute allowance relative to Cn^α is $|d/c|(a/b)^\alpha$. For the specified shells this is $(5/4)^{1-\alpha}$, greater than one for $\alpha < 1$. At $\alpha = 1/2$ it equals $\sqrt{5}/2$. This term prevents contraction of that absolute induction; it does not determine the size of the signed recurrence.

Problem 38. *Hint.* Solve the identity at $x = 2n$ and iterate until the argument of E is below one.

Solution. The recurrence is $E(n) = F(2n)+\frac{1}{4}E(n/2)$. With $J = \lfloor \log_2 n\rfloor$, its terminal value vanishes and

$$E(n) = \sum_{j=0}^{J} 4^{-j}F(2n/2^j), \qquad |E(n)| \le \frac{C\sqrt{2n}}{1-1/(4\sqrt{2})}.$$

Indeed, the termwise allowance has ratio $1/(4\sqrt{2}) < 1$. The arithmetic example in Problem 37 instead has ratio $\sqrt{5}/2 > 1$ at this scale. Here the specified coefficients permit the absolute induction; those arithmetic coefficients do not.

Part IV. Arithmetic energy at growing scales

Problem 39. *Hint.* The equal-digit positions are $0, 6, 12, 18, 24$, and the address multiplier is -4 modulo 25.

Solution. The direct calculations give

A	0	1	5	12
a	0	21	5	2
$C_5(a)$	0	1	1	2
$F_5(a)$	0	$-16/5$	0	$8/5$
$B_4(A/25)$	0	$-16/5$	0	$8/5$

For example, at $a = 21$ the five floor differences are $(0, 0, 0, 0, 1)$; at $a = 2$ they are $(0, 0, 1, 0, 1)$. The multiplier -4 is invertible modulo twenty-five, so every address appears exactly once. Lemma 16.1 identifies each reading after this permutation, and summing their squares preserves the whole energy. The equality is a relabeling of the table.

Problem 40. *Hint.* Only $k = 3$ has a nonzero remainder when four is divided by $k = 2, 3, 4$.

Solution. Only $1/2$ is a sampled reset. Its resetting frequencies are two and four, so the point contribution is $2^2/25 = 4/25$. For $k = 3$, the incomplete block has one clock and $M_3 = 1$. The readings $T_{3,1}(1), T_{3,1}(2)$ are $-1/3, 1/3$, while the shifted phase readings are $1/3, -1/3$. Hence $\mathcal{A}_5^{\mathrm{half}} = -2/9$. Since $H_2 = 3/2$ and $H_4 = 25/12$,

$$D_{25,1/2}[B_4] = \frac{4}{25} - \frac{8}{225} - \frac{1}{1125} = \frac{139}{1125}.$$

The difference from the point contribution is $-41/1125$. The shifted boundary is visible, but its contribution alone does not give the finite energy defect.

Problem 41. *Hint.* A primitive difference gives the interval sum. For the weighted sum, retain the actual weight at each of the eight addresses.

Solution. The primitive values are

$$(0, -1, 1, 2, 1, -1, 0, 0, 0).$$

Its range has width three; the interval $2 \le n \le 3$ attains that value. The unweighted period sum is zero, whereas

$$\sum_{n=1}^{8} w_n H_1(n) = -\frac{1}{2} + \frac{1}{3} + \frac{1}{12} - \frac{1}{20} - \frac{1}{15} + \frac{1}{42} = -\frac{37}{210}.$$

A constant partner therefore need not have zero weighted interaction with a completed cycle. Summation by parts must retain variation of the observation weights, as well as any variation of the partner.

Problem 42. *Hint.* There are two summations. The odd positive integers through M sum to $\lceil M/2 \rceil^2$.

Solution. Keeping the factor two in the mixed energy gives

$$|W_S(M)| \le 2(2S)\frac{129}{S^2}\lceil M/2 \rceil^2 = \frac{516}{S}\lceil M/2 \rceil^2.$$

At $S = 256$, the respective allowances are 129 and 516. For fixed $0 < \alpha < 1$, the condition $M \le S/8$ holds at all sufficiently large scales. The allowance has order $S^{2\alpha-1}$, so it is bounded for $\alpha \le 1/2$. Including every completed cycle means $M = \lfloor S/8 \rfloor$ and gives an $O(S)$ allowance. The count of changed families and the sum of the old cycle lengths consume the pairwise saving. This estimates the absolute sum; it is not a lower bound on the signed work.

Problem 43. *Hint.* The old field is -1 on odd addresses and zero on even addresses. The odd native masses sum to $\log 2$.

Solution. The new field is zero on odd addresses and one on even addresses. Only address three contributes to the removed shell. The three terms in the moving-field identity are

$$-\frac{1}{12}, \qquad -2\left(\log 2 - \frac{1}{2} - \frac{1}{12}\right), \qquad \frac{1}{5}.$$

Their sum is $77/60 - 2\log 2$. Directly, $E_2[p] = \log 2 - 1/2$ and $E_4[q] = 47/60 - \log 2$, which give the same difference. The identity $\sum_{n \text{ odd}} w_n = \log 2$ follows from the alternating harmonic series, or the two-clock integral in Section 9.

For the finite calculation, the difference of squares on $n > 4$ is $(-1)^n$. Thus

$$H(10) = -\frac{1}{12} - \frac{1}{30} + \frac{1}{42} - \frac{1}{56} + \frac{1}{72} - \frac{1}{90} + \frac{1}{110} = -\frac{137}{1386}.$$

Both field envelopes are one. Localization gives the complete enclosure

$$E_4[q] - E_2[p] \in \left[H(10) - \frac{1}{11},\ H(10) + \frac{1}{11}\right] = \left[-\frac{263}{1386}, -\frac{1}{126}\right].$$

Its upper endpoint is negative. This establishes the sign using a finite calculation and a proved tail allowance, without approximating $\log 2$ or omitting the parent shell.

Problem 44. *Hint.* Use the later-family formula to find its first two readings. Two independent annihilators suffice because the later-family space has dimension two.

Solution. The later-family matrix and a second annihilating row are

$$U = \begin{pmatrix} -3/4 & 0 \\ -9/20 & 0 \\ -1/10 & -1/4 \\ 3/10 & 1/28 \end{pmatrix}, \qquad J = (3, -5, 0, 0).$$

Both LU and JU vanish. The rows are independent, and U has rank two. Thus, for A with rows L, J, one has $\ker A = V$. The separate retained readings are $L(v) = 0$, $J(v) = -5$. With $W = \operatorname{diag}(1/30, 1/42, 1/56, 1/72)$, the joint matrix is

$$G = AW^{-1}A^T = \begin{pmatrix} 395136 & 21840 \\ 21840 & 1320 \end{pmatrix}, \qquad \det G = 44593920.$$

Proposition 8.4 therefore gives the attained minimum

$$(0, -5)G^{-1}(0, -5)^T = \frac{25 \cdot 395136}{44593920} = \frac{35}{158}.$$

The single functional L gives only the zero lower bound. The second reading detects inherited energy outside the later-family space; the joint matrix accounts for the overlap of the two readings. This test vector illustrates completeness of the invariant set at one finite scale.

Glossary

The entries give the meaning of the notation, a small example or useful distinction, and a route back to its calculation. Terms such as measure, primitive and total variation have their usual mathematical meanings. Terms such as collision amplitude and parent shell name their particular roles in the energy account.

Activation cutoff. The boundary selecting which integer addresses are observed. With cutoff S, the native tail admits $n > S$, beginning at $S + 1$. Moving the cutoff to $T > S$ removes the old readings on $S < n \leq T$. It changes the observation even if the field is unchanged. Activation at $S + 1$ admits the whole reading there, which need not be a clock reset (Section 9).

Address. A location where a field has a specified reading, such as a point of the periodic interval or an integer index. The observation determines how much mass that location receives. A grid address has mass $1/q$; an integer address in the native tail has mass $1/[n(n+1)]$. Under ordinary integration a single address has zero mass (Section 2).

Admissible periodic field. A specified field that is piecewise C^2 on finitely many arcs of the periodic interval. On each closed piece, its value and first two derivatives extend continuously. The actual readings at boundaries are assigned separately. A field that vanishes except at one address is admissible when that address is included in the boundary set; admissibility does not require continuity across a boundary (Section 3).

Atom. In the observations used here, an address carrying positive mass by itself. If $\rho(\{b\}) = m > 0$, its contribution to energy is $m|f(b)|^2$. The symbol δ_b denotes unit mass concentrated at b. Thus the equally weighted grid is the measure $q^{-1} \sum_{b \in X_{q,\theta}} \delta_b$. Its atoms explain why changing one reading can change sampled energy while leaving the ordinary integral unchanged (Section 2).

Bernoulli kernels. The fixed periodic functions $\psi(t) = \{t\} - 1/2$ away from integers, with $\psi(k) = 0$, and $\widetilde{B}_2(t) = \{t\}^2 - \{t\} + 1/6$. In boundary decomposition they weight value jumps, slope jumps and regular curvature. Their argument $qb - \theta$ records the location of a boundary within a sample cell. The first kernel has a jump and a specified midpoint reading; the second is continuous (Sections 3 and 4).

Boundary data. The actual reading, the two one-sided traces and the derivative traces at a boundary. These are distinct data. For the opening sawtooth at zero, the left trace is 1, the right trace is -1 and the reading is 0. For energy calculations, first form $h = |f|^2$ and its boundary data. The remaining input to the decomposition is h''_{reg} between boundaries (Sections 3 and 4).

Boundary localization. Restricting the effect of a boundary change to the readings it crosses. In a finite register, changing J columns costs the incidences in those columns, even when the rest of the register is large. This gives a budget before a smoothing width is chosen (Section 13).

Carrier signal. The sequence $u(n)$ modulated by an envelope in a moving-window field. The alternating carrier $(-1)^n$ has bounded partial sums, so its interaction can be controlled by variation of the weighted partner. This signal is distinct from the carrier modulus of an arithmetic kernel (Section 10).

Carry count. A sum of floor differences recording integer crossings at specified addresses. In the base-three table, the equal-digit positions $0, 4, 8$ give differences $0, 0, 1$ when $a = 1$, hence $C_3(1) = 1$. Subtracting a/p centers the reading. The carry representation expresses the whole centered table as clock samples, making shared resets and inherited readings available to boundary decomposition (Section 16).

Chinese remainder theorem. The rule combining congruences into one residue class. Conditions modulo c_1, c_2 are compatible when their residues agree modulo (c_1, c_2); their common rows then form one class modulo $[c_1, c_2]$. A finite observation must still test whether that class meets its interval (Section 12).

Clock. A periodic fractional or sawtooth profile. The clock $\{u/j\}$ has period j, rises linearly between resets and reads zero at multiples of j. A centered clock $2c\psi(rx + \phi)$ has frequency r, phase ϕ and coefficient c; its jump is $-2c$. Its specified reset reading is the midpoint. The two conventions are related by point-value replacement (Sections 5 and 9).

Collision. A coincidence of component contributions at a common observed address. For real readings u, v, the combined square contains $2uv$ in addition to their separate squares. At a shared reset, the combined jump and its interaction with the inherited field record that collision. The mixed term can be positive, negative or zero; the coincidence alone does not determine its sign (Sections 2 and 5).

Collision amplitude. For centered clocks resetting at b, the combined coefficient $C_b = \sum_{j \in I_b} c_j$. Their total jump is $-2C_b$. Coefficients 1 and -1 cancel at a shared reset, whereas coefficients 1 and 1 reinforce. Taking the sum before its squared magnitude retains both possibilities. In a right-hand fractional-clock field, the analogous reset amplitude is the negative divisor sum of its source coefficients (Sections 5 and 9).

Collision matrix. The matrix $H = G^{\mathrm{cont}} - G^{\mathrm{grid}}$ storing all pairwise mixed defects of a chosen component family. For amplitudes z, the total defect is $z^* H z$. The diagonal stores individual defects; off-diagonal entries retain interactions. Although each Gram matrix is positive semidefinite, their difference can have either sign. A change of coordinates reorganizes these interactions without supplying an estimate for them (Section 8).

Collision register. A finite table recording the components incident to each observed row, with their amplitudes and orientations. Its row is $\Delta(r) = \sum_\xi \sigma_\xi z_\xi I_\xi(r)$. Squaring this combined reading retains the interaction of every pair; the incidence kernel records their actual shared observations (Section 12).

Compatibility. The existence of a common solution to arithmetic conditions before restricting the observation. The residues 4 modulo 7 and 1 modulo 8 are compatible, but their common class 25 modulo 56 misses $3 < r < 6$. Compatibility alone is not an observed collision (Section 12).

Completed family. In the divisor application, an old family whose source finishes before the taper begins. For odd m with $8m \le S$, its field is the unchanged periodic profile H_m. It has zero period mean and an interval-sum bound $3m$. Screening uses this bound to control one interaction by $129m/S^2$; summing over many families must still account for their number and sizes (Section 18).

Conserved mass. The additivity of an observation under subdivision. If children partition a parent of mass m, their masses sum to m. Copying the same reading c to those children preserves the energy $m|c|^2$ as well. Different child readings require their own weighted square. For the native observation, resolving the first two addresses gives masses $1/2, 1/6$ and remaining tail mass $1/3$; their sum is still one. Restricting to the tail selects a share of that mass (Sections 1.3 and 9.1).

Constrained minimum. The least energy obtainable by adding any field from a specified allowed space V to an inherited field f. Invariant readings give lower bounds for this minimum. A complete collection detects every direction outside V and determines the minimum through the joint matrix. A partial collection can miss energy, so a small lower bound need not mean that a small energy is attainable (Proposition 8.4).

Defect. The signed difference between the energies of one field under two observations. The periodic convention is $D_{q,\theta}[f] = \int_0^1 |f|^2\,dx - Q_{q,\theta}(|f|^2)$. A positive value means the integral is larger; a negative value means the samples carry more energy. The square is taken before averaging: $Q(|f|^2)$ generally differs from $|Qf|^2$. A defect is quadratic in the field and can be zero (Sections 2 and 3).

Dilation. Composition with an integer multiplication of the periodic argument, as in $f(x) \mapsto f(rx)$. Dilation preserves the continuous integral of a periodic profile but can change which addresses a finite grid visits. On a six-point grid, doubling the argument visits $0, 1/3, 2/3$ twice each. The greatest common divisor of the frequency and grid size records that repetition (Proposition 8.2).

Energy. Squared magnitude integrated against the observation, $E_\rho[f] = \int |f|^2\,d\rho$. It is nonnegative for a positive observation. With masses $1/2, 1/2$, readings $1, 3$ have energy 5. A sum of fields also contains mixed energy, so energies cannot ordinarily be added component by component. The energy is a mean square only when the observation has total mass one (Section 2).

Field. A function supplying a specified reading at each address. The field includes its assigned boundary values, even where changing a value would leave an ordinary integral unchanged. Separating the field from the observation makes two operations distinguishable: changing a reading and changing the mass assigned to it. A moving arithmetic family can change both at the same step (Section 2).

Fixed carrier. An arithmetic parameter retained while the observation cutoff changes. For $R_q(a, b)$ the carrier modulus is q; $\mathcal{A}_q(M)$ changes the square cutoff M without changing q. A bound at a smaller prime generally changes the carrier as well, so it cannot automatically be used for a shorter prefix of the old kernel (Section 14).

Fourier mode. A periodic oscillation $e^{2\pi ikx}$, with integer frequency k. Averaging it on the unshifted q-point grid gives zero unless q divides k, in which case the average is one. The boundary proof uses this selection rule after expressing the field's jumps in its Fourier coefficients. An offset changes the phase of a surviving mode, producing the shifted Bernoulli kernels (Theorem 4.1).

Grid offset. The parameter θ in the sample addresses $(a+\theta)/q$, for $0 \leq a < q$. It is measured in cells: $\theta = 1/2$ shifts the grid by $1/(2q)$, not by half the whole interval. A boundary b is sampled precisely when $qb - \theta$ is an integer. Changing the offset can therefore remove a point contribution even when the continuous energy is unchanged (Section 3).

Inherited interaction. The cross term $2\operatorname{Re} K_\rho(f,g)$ between an old field f and an addition g. It is measured under the observation used for the new combined field. At one real reading it is $2fg$: an addition of -1 to an inherited reading 1 gives cross term -2, which combines with the new square 1 to remove the old energy (Proposition 2.1).

Intersection kernel. The weighted observation shared by two components: $K_v(\xi,\eta) = \sum_r v(r) I_\xi(r) I_\eta(r)$. It includes the common residue, the actual interval and its row weights. These entries form a Gram matrix of incidence vectors (Section 12).

Invariant reading. A linear combination L of readings that vanishes on every allowed addition $h \in V$. Consequently $L(f+h) = L(f)$, and this retained value forces an energy lower bound. For example, a second difference vanishes on every affine addition. Several such readings may share addresses; their separate lower bounds cannot generally be added without accounting for the overlap (Theorem 8.3).

Joint matrix. The matrix $G = AW^{-1}A^*$ for independent invariant readings arranged as rows of A, with positive diagonal observation weights W. Its off-diagonal entries measure overlap between readings in the reciprocal weights. For retained values $b = Af$, the lower bound is $b^*G^{-1}b$. It is the constrained minimum when $\ker A = V$ (Proposition 8.4). The overlapping-second-difference problem (Problem 19) shows why separate bounds can double-count energy.

Jump. Right trace minus left trace, $[h]_b = h(b+) - h(b-)$. A fractional clock resetting from one to zero has jump -1. Its actual value at the boundary remains a separate choice: replacing that reading by $1/2$ changes no trace and hence no jump. For energy decomposition, the required jumps are those of $h = |f|^2$ and its first derivative (Section 3).

Localization. Expressing a complete energy change as a finite calculation plus a proved allowance for the omitted tails. The old tail energy supplies the lower allowance and the new tail energy the upper one. A finite calculation may have the wrong sign while the resulting interval remains valid. Localization becomes effective when field envelopes and remaining observation mass make that interval sufficiently short (Theorem 9.4).

Logarithmic prefix transform. The operation

$$\mathcal{K}(N) = -\sum_{d \le N} \mu(d) \log d \, \mathcal{A}(\lfloor N/d \rfloor)$$

on square prefixes of a fixed kernel. For a dilation-invariant kernel it equals $\sum_{m \le N} P(m)\Psi(N/m)$, combining primitive shells with the prime-power count (Theorem 14.2).

Midpoint reading. The arithmetic mean $(f(b-) + f(b+))/2$ of a field's two traces. The average of their squared magnitudes exceeds the squared midpoint by $|[f]_b|^2/4$. This produces a nonnegative point contribution at sampled boundaries. An actual right-hand reading requires a further replacement term, which includes interaction with the midpoint. Its correction need not be positive (Corollary 4.2 and Section 9).

Mixed energy. The form $K_\rho(f, g) = \int f\overline{g} \, d\rho$. Twice its real part is the cross term in $E_\rho[f + g]$. It is linear in the first field and conjugate-linear in the second, and interchanging the fields conjugates the result. These are sesquilinearity and Hermitian symmetry. Subtracting the mixed energies under two observations gives the mixed defect, which need not be positive (Sections 2 and 8).

Möbius coefficient. The coefficient $\mu(r)$ is zero when a prime square divides r. Otherwise $\mu(r) = (-1)^k$, where k counts the distinct prime factors, with $\mu(1) = 1$. Thus $\mu(3) = -1$, $\mu(15) = 1$ and $\mu(9) = 0$. These signed coefficients join the divisor families. Their divisor identity permits inversion between family data and reset data; their signs alone do not bound the resulting energy (Sections 14 and 18).

Modular inverse. An integer $\bar{a}$ satisfying $a\bar{a} \equiv 1 \pmod{k}$; it exists precisely when $\gcd(a, k) = 1$. For example, 25 is its own inverse modulo four, while 50 has no inverse modulo four. The half-cell calculation therefore separates odd and even moduli before making inverse substitutions. A bar in that calculation refers to the modulus of its summand (Section 17).

Native weights. The prescribed masses $w_n = 1/[n(n+1)] = 1/n - 1/(n+1)$. They are the reciprocal-cell integrals $\int_n^{n+1} u^{-2}\,du$, so the tail beyond S has total mass $1/(S+1)$. Each later address receives less mass, but every address in the infinite tail remains present. Summation by parts must retain the variation of these masses as well as that of the field (Section 9).

Observation. A finite positive measure specifying which addresses are read and with what mass. It covers ordinary integration, a finite weighted table and a summable infinite tail. The word measure records the assignment of mass, not an additional field. For an observation of total mass M, the constant field one has energy M. Changing the observation can change energy without changing a single reading (Section 2).

Operator norm. The largest ratio $\|Mv\|/\|v\|$ over nonzero input vectors, using the specified vector norm. It bounds every input direction. A prescribed quadratic reading $v^T Mv$ may be small even when the operator norm is large. The arithmetic interior construction makes that distinction explicit (Proposition 15.1).

Parent shell. The addresses $S < n \le T$ removed when a native cutoff moves from S to T. Their entire old energy enters the change with a minus sign. If the old field is a sum of families, the shell term contains their combined square and every cross term. Replacing it by a sum of family energies would alter the account (Proposition 9.3).

Point-value replacement. Changing assigned readings while retaining the surrounding profile. At an atom of mass m, replacement of $f(b)$ by $g(b)$ changes sampled energy by $m(|g(b)|^2 - |f(b)|^2)$. It changes an integral-minus-sample defect by the opposite amount when the integral is unchanged. At an unobserved address the correction is zero. This rule restores actual readings after a convenient midpoint calculation (Proposition 3.1).

Polarization. Recovering interaction from the energy of a sum. Here the basic expansion is $D[f+g] = D[f] + D[g] + 2\operatorname{Re} K(f,g)$, where K is the mixed defect. Real fields f and $-f$ illustrate its role: their sum has zero defect even when each separate defect is nonzero. The collision matrix organizes the same expansion for many components (Section 8).

Prime-counting function. Here the weighted prime-power count $\Psi(x) = \sum_{\ell^j \le x} \log \ell$, where ℓ runs over primes and $j \ge 1$. For example $\Psi(4) = 2\log 2 + \log 3$. The logarithmic transform gives a finite identity involving this quantity; estimating its error requires further control of the signed recovery (Section 14).

Primitive. A cumulative sum, playing the role of an antiderivative for a sequence. Anchored at a, it is $T_a(n) = \sum_{k=a}^{n} u(k)$ with $T_a(a-1) = 0$. For $u = (1, -1, 1, -1)$ its readings are $1, 0, 1, 0$. A small primitive records cancellation in u. Summation by parts pairs it with differences of the weighted partner, retaining endpoints (Section 7).

Primitive shell. The signed kernel sum over coprime pairs whose maximum coordinate is m. Each such pair has $\lfloor M/m \rfloor$ common multiples in a square cutoff M. Primitive means coprime coordinates; it does not mean a prime shell index, and differs from a cumulative primitive (Proposition 14.1).

Quadrature. Estimating an integral by a weighted sum of sampled readings. The periodic grid rule $Q_{q,\theta}$ is one such operation. Its discrepancy is linear in the sampled function; applying that discrepancy to $|f|^2$ gives the energy defect. A quadrature rate for a fixed function can depend on its jumps and curvature, which must be counted again when the field grows (Sections 4 and 11).

Quantitative recovery. Reconstructing a quantity while controlling the size of the error or bound carried through the reconstruction. A triangular identity can be exact even when an absolute-value induction enlarges its allowance. The prime-shell example has coefficient $\sqrt{5}/2 > 1$ at square-root scale (Section 15).

Reciprocal neutrality. The condition $\sum_j q(j)/j = 0$ for a finite fractional-clock source. Each clock has slope $1/j$ between resets, so this signed sum is the slope of their combined field. Neutrality makes that field constant between integer addresses, leaving all changes at resets. For $p(u) = \{u\} - 2\{u/2\}$, the source sum is $1 - 2/2 = 0$ (Section 9).

Reciprocal phase transfer. The identity $e(A\bar{r}/c) = e(-A\bar{c}/r)e(A/(rc))$ for coprime $r, c > 1$ and integer A, with $e(x) = \exp(2\pi i x)$. It changes the modulus of an inverse while retaining a reciprocal correction. That correction remains part of any subsequent phase estimate (Proposition 13.3).

Refinement. A specified change of resolution or scale. Refining a grid changes the sampling observation. Increasing the number of clocks changes the field as well; moving its activation cutoff changes the observed domain. Each change has its own term in the energy identity. Statements about refinement therefore specify which objects vary and which stay fixed (Sections 8 and 9). For unity, subdivision resolves the same total mass, while admitting new clocks enlarges the space available to represent the fixed target (Section 1.3).

Regular curvature. The ordinary second derivative h''_{reg} on the open pieces between boundaries. Value jumps and slope jumps are recorded separately in the boundary formula. For a piecewise linear field, the field itself has zero second derivative on each piece, but its squared magnitude generally has nonzero second derivative. Energy decomposition requires the latter, because its input is $h = |f|^2$ (Section 4).

Reset. An address where a fractional clock returns to the start of its cycle. The actual fractional-part reading is zero, its right-hand value. Clocks of lengths dividing n reset together at integer address n. Their signed coefficients are combined before forming the energy change. A reset of some clocks does not remove the inherited reading of all the others (Sections 5 and 9).

Residual. The terms remaining after a specified contribution has been separated from an identity. For a periodic defect, $R = D - P$ remains after the sampled point contribution P. For native work, the remainder after short-cycle screening includes the parent shell and unscreened tail interactions. A residual keeps its sign. Naming it does not imply that it is small or tends to zero (Sections 8 and 19). In approximation of unity, the residual is the field $r_N = \mathbf{1} - p_N$; its complete squared error is $E_0[r_N]$. The field and its scalar error have different roles (Section 9.1).

Screening. Control of an interaction by cancellation within an old completed cycle. Its field has zero mean, so a constant partner cancels over a full period. For a varying weighted partner, summation by parts uses the old field's bounded primitive and the partner's variation. The native estimate $129m/S^2$ controls each interaction; the sum also depends on how many cycle lengths m are included (Theorem 18.2).

Signed work. The change in observed energy under a specified modification of field and observation. For an addition g under a fixed observation it is $2\operatorname{Re} K_\rho(f, g) + E_\rho[g]$. A moving cutoff also contributes the removed parent energy. Keeping these terms joined permits negative interaction to offset the new square. Bounding their absolute values separately can lose that cancellation (Proposition 9.3).

Smoothing width. The number of boundary columns over which a sharp cutoff is replaced by a smooth transition. Wider transitions change more readings and have smaller derivative norms. Balancing costs J and $p/\sqrt{J}$ selects $J \asymp p^{2/3}$ under the stated smooth-row hypothesis (Proposition 13.2).

Source. A coefficient table for a clock field. Each entry $q(j)$ gives the amplitude of the clock $\{u/j\}$, whose period is j. This $q(j)$ is distinct from the earlier scalar grid size q. Its reciprocal sum controls the slope, and its divisor sum $c(n) = \sum_{j|n} q(j)$ controls the simultaneous reset. Sources therefore convert coefficient data into local field changes (Section 9).

Tail. All readings beyond a chosen finite endpoint. For native weights, the tail beyond N has mass $1/(N+1)$. If $|p(n)| \leq A$ there, its energy is at most $A^2/(N+1)$. When A depends on an outside scale, the endpoint must grow sufficiently fast to control that dependence. A finite head together with this allowance encloses the full observation (Theorem 9.4).

Taper. A factor that reduces a source profile before its terminal cutoff. The divisor application uses τ_S, equal to one through S, decreasing linearly to zero at $2S$, and zero beyond. Taking the difference of a tapered profile also differentiates the taper discretely. Its resulting coefficients, including the terminal one, must remain in the source (Section 18).

Total variation. The supremum, over ordered partitions, of the sum of absolute successive changes in a function. For a monotone function it is the absolute endpoint difference: x^{-2} has variation $3/4$ on $[1, 2]$. In weighted primitive estimates, variation measures how strongly the partner can disturb the carrier's cancellation. Both changing weights and changing field values contribute (Section 10).

Trace. A one-sided limiting reading, written $f(b-)$ or $f(b+)$. At the periodic origin, the left trace comes from addresses near one and the right trace from addresses near zero. The actual value $f(b)$ may equal either trace, their midpoint or another assigned value. Traces describe the surrounding profile; the observation determines whether the assigned point reading also contributes (Section 3).

Uniform estimate. A bound with constants independent of the varying field or scale over the stated family. The assertion $|D_q[f]| \leq C_f/q$ for each fixed f does not by itself control $D_q[f_q]$: the constants C_{f_q} may grow with q. A uniform proof bounds the relevant amplitudes, jumps, primitive sizes and variation across the whole family before taking the limit (Section 11).

Unity. The constant field $\mathbf{1}(x) = 1$, used as a fixed target through changing resolutions and approximation spaces. Under an observation of mass one its energy is one. Under the native tail observation it has energy $E_S[\mathbf{1}] = 1/(S+1)$, the mass actually retained. An approximation p_N leaves residual $\mathbf{1} - p_N$. For example, twice the period-two fractional clock equals unity at odd addresses and leaves the even addresses unresolved. Their complete error mass is $1 - \log 2$. Unity, total mass and the energy of a general field are distinguished in Sections 1.3 and 9.1.

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