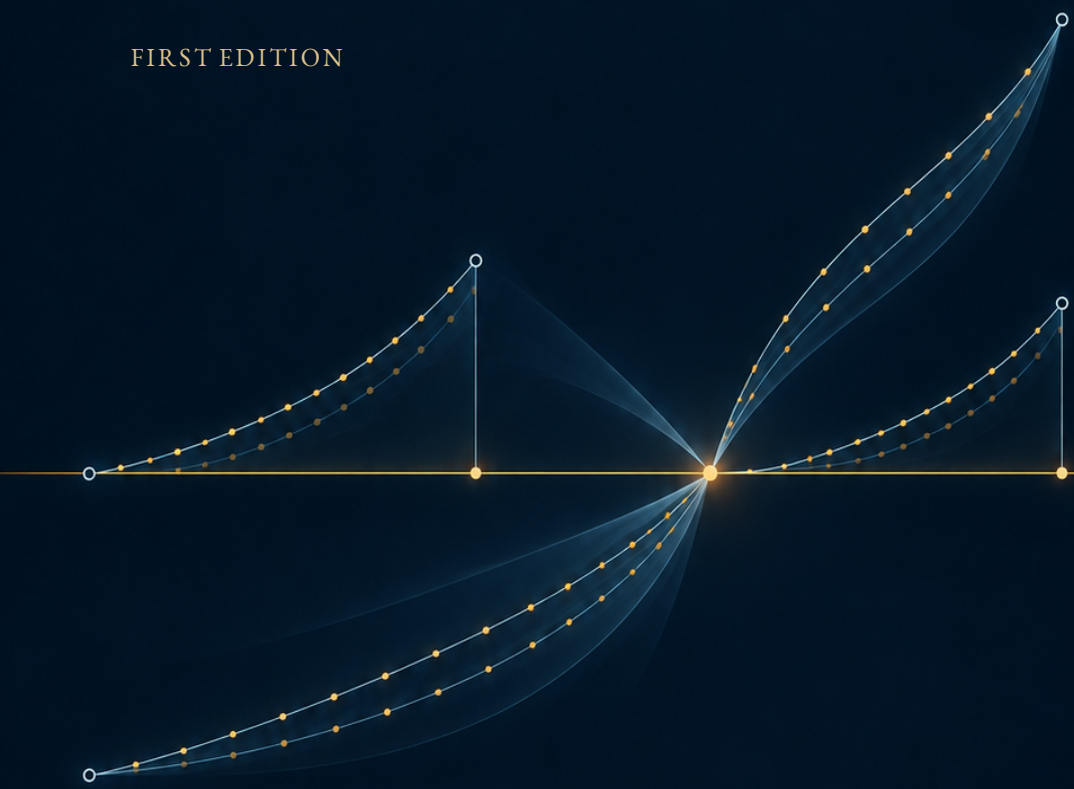
# COLLISION CALCULUS

## Boundary Readings, Energy and Refinement

FIRST EDITION



Alexander S. Petty

# Collision Calculus

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First Edition

Boston Agile Labs Press

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**Collision Calculus**
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An early digital version appeared in October 2026.

*For Cary, Griffin, and Killian,*

*for the life we share*
*and the wonder still before us.*

*"The rules of the finite hold in the infinite … and conversely, the rules of the infinite hold in the finite."*

Gottfried Wilhelm Leibniz

To Pierre Varignon, 2 February 1702
Translated from the French, excerpted

*Leibnizens mathematische Schriften*, edited by C. I. Gerhardt, vol. IV, pp. 93–94.

## Preface

An integral and a finite table of readings can describe the same field yet give different answers for its energy. A single boundary can cause the difference. So can interactions that accumulate as the field and its observation grow.

I call these calculations *collision calculus* because they follow what happens when arithmetic components meet. The name began with agreements between digits and a question about what survives as the surrounding arithmetic changes.

The alignment-deficit lattice gave that question a musical form. Counts of repeating digits recover the octave, fifth and major third, with an exact correction at every finite scale. That connection between the whole and its finite readings runs through this book.

Here, the meetings occur at boundaries. Several components can reset at the same point. Their signed amplitudes add before the square is taken. They can reinforce one another or cancel, and their interaction with the rest of the field stays in the calculation.

A single point has no weight in an integral. A finite observation can give it weight. The same boundary belongs to both descriptions, but it is not read in the same way. The mathematics here asks how our understanding of a whole can grow when the relation between its continuous structure and its finite readings becomes part of the calculation.

Some of the methods are classical, including summation by parts and quadratic forms. Other constructions and estimates were developed in the arithmetic work. Their sources are identified throughout and collected in Section 31.

The opening follows a digit count into the cubic energy law and its corrections. A line and three readings then make the boundary question elementary. Worked calculations and solved problems develop methods that can be carried into another problem, with the observation, interactions and remaining estimate kept in view.

An early digital version appeared in October 2026. This First Edition adds methods from the continuing work, develops the arithmetic opening and revises the exposition throughout. Figures now accompany the harder arguments as well as the first examples. Reader feedback has helped make the explanations more concrete and the final applications easier to follow. I am grateful for that care.

## Reader's guide

Calculus, finite sums and elementary linear algebra provide the starting point. Most of the development is suited to advanced undergraduate or beginning graduate study, including independent study. The proofs draw increasingly on real analysis, Fourier analysis and elementary number theory. The final arithmetic applications require more analytic background than the opening examples.

The opening essay, Finite readings and the continuous whole, sets out the question behind the calculations. Begin the arithmetic with the digit count in Section 1. It follows its finite invariant into the cubic energy law, then distinguishes the continuous secondary term from the sampling defect. The asymptotic results are stated with their sources; their long analytic proofs are not prerequisites for the first calculations.

In Section 2, work out the three squares before reading the general definitions. That simple field returns in the boundary calculation, so each new piece of notation has an example already attached to it.

In Parts I and II, a finite positive measure can initially be read as a list of nonnegative weights. If Fourier series are unfamiliar, use the hypotheses and formula of Theorem 5.1 in the examples, then return to its proof. The calculation itself needs the actual reading, two traces and slope traces at each boundary.

The moving window in Section 11 and the overlapping pulses in Section 12.2 bring the first methods together. They combine several operations and obtain the whole answer. The native two-clock tail in Section 10 does the same for an infinite observation. If you can account for the mixed term and the endpoints in those three examples, the arithmetic applications have a concrete starting point.

Part 3 begins with the alignment-deficit lattice. Its digit count recovers the octave, fifth and major third with explicit finite corrections. The binary-field realization connects that count to the earlier energy definitions. Familiarity with congruences and divisor sums is useful here. Congruence registers, primitive shells and source reconstruction follow. Congruence intersections retain the observed rows. Divisor inversion then groups repeated readings and reconstructs their sources. The final recovery calculation shows why an inverse formula needs a quantitative cost before it can carry an error estimate.

Part 4 develops the additional methods through small finite examples, using orthogonal projections, matrix norms and finite Fourier

modes. Begin with the two source columns and their weighted Gram. The same example returns for the adjoint receiver. Between them, the four-address cycle supplies the difference, return and refinement calculations. A new example always specifies its source and observation; equal-sized tables need not represent the same experiment.

For a short problem route, the following choices test distinct steps. The remaining problems give additional practice and changes of data.

| Operation to practice | Problems |
|---|---|
| Readings, traces and interaction | 1, 6, 27. |
| Complete weighted observations | 15, 28. |
| Arithmetic coordinates | 29, 33, 36, 39. |
| Measured recovery | 43, 45. |
| Returns and refinement | 46, 47. |
| Packets and signed readings | 48, 49. |

Part 5 develops two sustained applications. The carry calculation changes the observation of a fixed clock field at each prime scale. Its half-cell outcome and finite examples precede the longer parity argument. That proof uses additional analytic estimates identified at its entrance and is best approached with a graduate background in analysis. Its source identities and finite examples can be read first on their own. The native application changes the field and its cutoff together. Its energy question is stated as soon as the source is defined, then followed through three sections to the remaining uniform bound. The closing two-clock calculation shows how a finite head and its tail allowance can decide the sign of a complete energy change. Unity and Plurality closes the main text by returning to the relation between a finite calculation and what it claims about the whole.

The hints and solutions follow the problem order and link back to their questions. The operation table in Section 8 gives entry points for a calculation. The symbol table, glossary and subject index support a return to the book after time away from it. Section 31 identifies the sources of the methods.

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## Finite readings and the continuous whole

Counting begins by making distinctions. Measurement gives those distinctions a unit and a place. Choose a unit of length, mark its integer multiples, then subdivide the intervals. Numbers give the grid its addresses. An observation specifies the readings taken there and the weight each carries.

Calculus lets us study what happens under arbitrarily fine subdivision. Finite arithmetic retains the individual readings and their exact relations. These descriptions bring different information into view. An integral can ignore a single point. A limiting law can leave a finite correction unresolved. A table can give every entry correctly while leaving the behavior beyond its scale unknown. Reconciling the descriptions requires us to account for what each leaves out.

What does a finite description reveal about a continuous whole, and what remains unresolved? Collision calculus examines this question through energy, the squared magnitude of a field, summed or integrated with specified weights. The field supplies the values; the observation specifies how they are read. A fuller account includes how the two energies differ and how that difference changes with scale.

Digital sound and images make the question familiar. A recording stores a sequence of readings; an image resolves light into pixel values. Classical sampling theory gives conditions under which an ideal sequence of exact samples determines a signal whose frequencies are bounded [25]. A finite recording with rounded values adds questions of duration and precision. Fidelity depends on the signal, the observation and the information used to reconstruct it.

The energy question becomes visible at a discontinuity. Imagine several periodic signals rising and resetting together. Just before the reset, their amplitudes reinforce. At the reset itself, the recorded value may be zero. An integral is unchanged by the value assigned to that single point. A finite average gives the point a share of the total. Both calculations can be correct and still disagree about the energy.

On an equally weighted grid, adding sample points reduces the weight of any one reading. The signal may grow at the same time. More components may meet at a reset, and their combined amplitude is squared before it enters the energy. The rising amplitude can compensate for the shrinking sample weight. A discrepancy can survive even as the grid becomes arbitrarily fine.

Such a discrepancy can change a conclusion. It can reverse the apparent sign of an energy change or conceal the effect of adding a component. A bound proved for a fixed field must expose its dependence on the field before it can be used for a growing family.

The arithmetic in this book makes these questions concrete. Every digit, reset and weight is specified. Multiplication moves fractions through digit bins, producing a table of agreements. The selected digit words then define a carry table whose centered readings have an energy. Its leading growth follows a cubic law. The corrections describe structure that the leading law alone does not resolve, including the different ways a finite grid and an integral read a common reset.

We call the continuous energy minus the sampled energy the *sampling defect*. A positive defect means that the finite observation records less energy; a negative one means it records more. Its size, its sources and its dependence on scale are mathematical questions of their own. A persistent boundary contribution can explain part of it. Establishing the limit of the whole defect also requires control of the remaining signed contributions.

Boundary formulas locate the contributions to the discrepancy. Quadratic forms keep the interactions between components in the same account. Summation by parts follows cancellation when the observation changes. Together with reconstruction and tail bounds, these methods let us determine what a finite record says about the whole and what remains beyond its reach.

The intuition reaches further. Is continuity fundamental to nature, or does it arise from a finer structure? Planck time combines the quantum, gravitational and relativistic constants into a natural time scale [11]. It is not an established smallest tick of time. A limit to physical resolution and a discrete structure of spacetime are distinct possibilities [7]. Whether there is a finest flicker beneath continuous change remains a physical question. The work here examines the mathematical relation between finite and continuous descriptions in specified arithmetic and analytic models. It asks what can be preserved, recovered or lost when we pass between them.

We begin with six fractions and two bases. Every agreement can be checked by hand. Changing only the base changes which pairs agree, giving our first view of a structure through the rule used to read it.

Part 1

# Collision energy and boundary readings

## 1. Collision tables and the cubic law

The first question is how a change in the reading rule changes a finite count. We can see the effect before introducing energy or passing to a limit.

Take the six fractions $1/7, 2/7, \dots, 6/7$. In base three, a fraction's first digit after the point is zero in the first third of the unit interval, one in the middle third, and two in the last third. Our six fractions therefore begin with $0, 0, 1, 1, 2, 2$. These are the three *digit bins*. Multiply each fraction by two, discard its integer part, and read the first digit again. Some digits agree with their original readings. Others change. An agreement is a *digit collision*.

For example, $1/7$ and $2/7$ both begin with zero in base three, so the fraction $1/7$ gives a collision under doubling. The fraction $2/7$ does not. Its doubled value $4/7$ begins with one. Doubling $4/7$ gives $8/7$; discarding the whole unit leaves $1/7$, so its digit changes from one to zero.

1.1. **Reading a collision table.** A collision table collects these comparisons. Each column specifies an original remainder $r$. Each row specifies a multiplier $g$. The entry is one when $r/7$ and the fractional part of $gr/7$ have the same first digit, and zero when they do not. The row sum counts agreements for that multiplier.

Figure 1 gives the complete table.

| | original remainder $r$ | | | | | | row sum |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | agreements |
| $g = 1$ | 1 | 1 | 1 | 1 | 1 | 1 | 6 |
| $g = 2$ | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| $g = 3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $g = 4$ | 0 | 1 | 0 | 0 | 1 | 0 | 2 |
| $g = 5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $g = 6$ | 0 | 0 | 1 | 1 | 0 | 0 | 2 |

a filled cell records equal first digits
in the two compared fractions

FIGURE 1. Base three, modulus seven. Column $r$ begins with the fraction $r/7$; row $g$ compares it with $[gr]_7/7$. A one records a digit agreement. The outlined doubling row has two agreements, at remainders one and six. The row sum keeps the count but loses the locations of those agreements.

It includes all six multipliers and all six nonzero remainders. The row $g = 2$ records the doubling experiment. Its two ones occur at $r = 1$ and $r = 6$. The row $g = 1$ has six ones because every fraction is compared with itself. The row $g = 3$ has no ones. None of those six first digits survives a shift by one base-three digit.

The table is a list of outcomes, not a probability model. Every cell comes from a specified comparison. A row sum records how often its comparison succeeds, while the positions of the ones record where it succeeds. Those are different amounts of information.

1.2. **Changing the base.** Keep the same six fractions and read them in base two. The digit is zero below $1/2$ and one above it, so the first digits are now $0, 0, 0, 1, 1, 1$. The comparisons give the following binary collision table. Again, a one records agreement and a zero records disagreement.

| $g\backslash r$ | 1 | 2 | 3 | 4 | 5 | 6 | Count |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 6 |
| 2 | 1 | 0 | 0 | 0 | 0 | 1 | 2 |
| 3 | 1 | 0 | 1 | 1 | 0 | 1 | 4 |
| 4 | 0 | 1 | 0 | 0 | 1 | 0 | 2 |
| 5 | 0 | 1 | 1 | 1 | 1 | 0 | 4 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Read the row $g = 3$ one column at a time. The original remainders $1, 2, 3, 4, 5, 6$ become $3, 6, 2, 5, 1, 4$ after multiplication by three and reduction modulo seven. Their binary first digits are $0, 1, 0, 1, 0, 1$. Comparing with $0, 0, 0, 1, 1, 1$ gives $1, 0, 1, 1, 0, 1$, hence four collisions.

For one cell, take $r = 3$. The fraction $3/7$ lies below $1/2$, and its tripled value $9/7$ leaves $2/7$, also below $1/2$. Both begin with zero in base two. In base three, $3/7$ begins with one and $2/7$ with zero. The same pair collides in one base and not in the other. The fractions and multiplication are unchanged; the partition used to read a digit has changed.

1.3. **The collision invariant.** Write $[x]_N$ for the least nonnegative residue of $x$ modulo $N$. Multiplying a fraction $r/N$ by $g$ and discarding whole units leaves $[gr]_N/N$. Its first base-$b$ digit is obtained by multiplying by $b$ and taking the integer part. We write $\lfloor x \rfloor$ for the greatest integer at most $x$.

Fix a base $b \geq 2$ and a lag $\ell \geq 1$. For an integer modulus $N > b^{\ell+1}$ coprime to $b$, write

$$\delta_N(r) = \left\lfloor \frac{br}{N} \right\rfloor, \qquad C_N(b^\ell) = \#\{1 \leq r < N : \delta_N(r) = \delta_N([b^\ell r]_N)\}.$$

The count compares digit positions separated by $\ell$ steps. Its deviation from the baseline count is

$$S_\ell(N) = C_N(b^\ell) - \left\lfloor \frac{N-1}{b} \right\rfloor.$$

The finite-determination theorem states that $S_\ell(N)$ depends only on $N$ modulo $b^{\ell+1}$ [15]. Primality of $N$ is not required. At base three and lag one, the moduli $11, 20, 29$ all have residue two modulo nine. Their collision counts are $4, 7, 10$, their baselines are $3, 6, 9$, and each deviation is one. The count grows while that remainder stays fixed.

The table uses the complete set of nonzero remainders. A single reciprocal expansion need not visit them all. This distinction lets the invariant describe the whole finite ensemble without assuming that the base generates one complete orbit.

The modulus $b^{\ell+1}$ has a concrete origin. Divide the remainder interval into that many slices and put $n = \lfloor b^{\ell+1} r/N \rfloor$. The two compared digits are the leading and trailing base-$b$ digits of $n$. At lag one, the equal-digit words are therefore

$$G_b = \{(b+1)d : 0 \le d < b\}.$$

In base three these are $00, 11, 22$, at positions $0, 4, 8$ among the nine two-digit words. A digit agreement has become a finite geometric selection.

The reduction for general weighted digit rules, with their bulk and endpoint terms retained, is developed in *Carry Boundaries and Bernoulli Spectra in Long Division* [14].

1.4. **The energy of the complete carry table.** The equal-digit selection leads to a second table, which records boundary crossings. Let the base be an odd prime $p$. Its selected positions $G_p$ lie on a circle of $p^2$ addresses. A row chooses a multiplier $a$, and a column chooses a selected position $n$. Multiply $n$ by $a$, reduce it to the circle, then advance it by $a$. Enter one if the step crosses the join and zero otherwise. This is a *carry table*. Its floor formula is

$$\left\lfloor \frac{(n+1)a}{p^2} \right\rfloor - \left\lfloor \frac{na}{p^2} \right\rfloor.$$

Since $0 \le a < p^2$, each difference is zero or one. At base three and $a = 2$, the selected positions $0, 4, 8$ become $0, 8, 7$ on the nine-address circle. Advancing by two crosses the join from eight and from seven. That row has two carries.

A step of length $a$ occupies the fraction $a/p^2$ of the circle. The proportional allowance for $p$ selected positions is therefore $p(a/p^2) = a/p$. It is a geometric reference level, without a randomness assumption. Subtract it from the row count to record the excess or shortfall,

$$C_p(a) = \sum_{n \in G_p} \left( \left\lfloor \frac{(n+1)a}{p^2} \right\rfloor - \left\lfloor \frac{na}{p^2} \right\rfloor \right), \qquad F_p(a) = C_p(a) - \frac{a}{p}.$$

Here $C_p(a)$ is the carry count at base $p$, distinct from the digit-agreement count $C_N(b^\ell)$ above. The finite slice calculation has converted digit agreement into boundary data. The two tables have different columns and entries; their connection is the selected set of equal-digit words.

The full base-three carry table displays the centering row by row.

| | Selected word | | | Count | Allowance | Deviation |
|---|---|---|---|---|---|---|
| $a$ | 00 | 11 | 22 | $C_3(a)$ | $a/3$ | $F_3(a)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1/3 | 2/3 |
| 2 | 0 | 1 | 1 | 2 | 2/3 | 4/3 |
| 3 | 0 | 0 | 1 | 1 | 1 | 0 |
| 4 | 0 | 1 | 1 | 2 | 4/3 | 2/3 |
| 5 | 0 | 0 | 1 | 1 | 5/3 | −2/3 |
| 6 | 0 | 1 | 1 | 2 | 2 | 0 |
| 7 | 0 | 0 | 1 | 1 | 7/3 | −4/3 |
| 8 | 0 | 1 | 1 | 2 | 8/3 | −2/3 |

The deviations add to zero, although most readings differ from their allowances. Their signed sum would hide the size of the fluctuations. Squaring each deviation before adding gives the collision energy

$$E_p = \sum_{a=0}^{p^2-1} F_p(a)^2.$$

The base-three table has four deviations of magnitude $2/3$ and two of magnitude $4/3$, so $E_3 = 4(2/3)^2 + 2(4/3)^2 = 16/3$. The energy records the size of the fluctuations without allowing positive and negative readings to cancel before they are squared. Interactions between the components of each reading remain inside its square.

As the odd prime base grows, the cubic law gives [17]

$$E_p = p^3 + O\big(p^2(\log p)^2\big).$$

Thus $E_p/p^3$ tends to one. The error bound permits an absolute discrepancy that grows, even though its proportion of $p^3$ tends to zero. The law describes a limit, not an approximation promised to be close at base three.

There are $p^2$ addresses, and the mean squared reading has leading size $p$. Their product gives the cubic scale. The theorem establishes that growth for the specified table and its centering.

1.5. **Structure inside the remainder.** The notation $O(A_p)$ means that the absolute remainder is at most a constant times $A_p$ for all sufficiently large $p$. It bounds size without describing structure. The secondary-term result identifies its leading part [20],

$$E_p = p^3 - \frac{p^2}{\pi^2}(\log p)^2 + o\big(p^2(\log p)^2\big).$$

Here log is the natural logarithm, and $o(A_p)$ means a quantity whose ratio to $A_p$ tends to zero. The leading cubic approximation eventually overestimates the energy. The amount by which the energy falls below that approximation has a definite coefficient and a scale of its own.

To locate that correction, use the periodic sawtooth

$$\psi(x) = \begin{cases} \{x\} - \frac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases} \qquad B_N(x) = 2\sum_{j=1}^{N} \psi(jx).$$

The notation $\{x\}$ means the fractional part of $x$. Each component rises and resets, and the assigned reading at a reset is its midpoint. Section 25 proves that the centered carry table is an exact permutation of the readings of $B_{p-1}$ on the $p^2$ grid. Consequently its mean squared reading is

$$\frac{E_p}{p^2} = \frac{1}{p^2}\sum_{a=0}^{p^2-1} B_{p-1}(a/p^2)^2.$$

The continuous energy of the same field is

$$J(N) = \int_0^1 B_N(x)^2\,dx = \frac{1}{3}\sum_{r,s\le N} \frac{\gcd(r,s)^2}{rs}.$$

Here $\gcd(r,s)$ is the greatest common divisor. Each clock has continuous energy $1/3$, so the $N$ individual squares contribute $N/3$. Their mixed products supply the remaining leading energy. The finite cubic theorem retains the corresponding division of its leading mass into one third from individual squares and two thirds from mixed products [17].

A complementary description of this capacity by weighted remainder ages, and the resulting profinite dynamics, is developed in *Conservation and Profinite Dynamics in Digit-Collision Energy* [16].

That quadratic form already contains the logarithmic correction. The continuous secondary-term theorem gives [20]

$$J(N) = N - \frac{1}{\pi^2}(\log N)^2 + O\big((\log N)^{5/3}\big).$$

Transferring its coefficient to the finite table requires a separate bound for the sampling difference. The two calculations observe the same field, but they need not give the same energy.

1.6. **The common reset.** Define the sampling defect by

$$S_p = J(p-1) - \frac{E_p}{p^2}.$$

For every $\varepsilon > 0$, the unit-defect theorem gives, through odd primes [21],

$$S_p = 1 + O_\varepsilon(p^{-1/4+\varepsilon}).$$

All $p-1$ clocks reset at zero. The field reads zero there, whereas the squared profile approaches $(p-1)^2$ from either side. An integral ignores the assigned value at that single point. The finite grid gives it mass $1/p^2$. Restoring the limiting squared reading therefore adds $(p-1)^2/p^2$ to the sampled mean, a contribution tending to one. The theorem also controls the combined remainder from the other boundaries. The common reset alone would not prove the limit.

At $p = 5$, the continuous energy is $23/9$ and the sampled energy is $48/25$. Their difference is $143/225$. Restoring the common reset adds $16/25$ to the sample average and leaves the small signed difference $-1/225$. Section 25 returns to this calculation after developing the boundary formula.

The logarithmic secondary term and the unit sampling defect concern different comparisons. The first refines the continuous energy's leading growth. The second measures how a finite observation reads that growing field. The continuous remainder must still be resolved before either can be called a complete finer expansion of $E_p$.

This is the question behind the calculus. How does a finite calculation retain the energy of the continuous field it describes? The cubic law gives the scale. Its corrections expose the arithmetic of

that contact. To learn how to calculate them, begin with one line and three readings.

## 2. Continuous and sampled energy

To isolate the boundary effect, hold the field fixed and begin with a single periodic ramp. An integral and a sample average are two ways of observing that field. They need not give the same answer. A sampling error can reverse which signal appears to have more energy. It can also change the predicted effect of adding a component. Before estimating that error, we need to know what each observation reads.

Here energy means squared magnitude, integrated or averaged. For a real reading $u$ it is $u^2$; for a complex reading it is $|u|^2$. Squaring makes interaction unavoidable. Two contributions of size one give an energy of four when they reinforce and zero when they cancel. Their separate energies add to two in either case.

The first boundary calculation needs only a straight line, three sample points and the sum of their squares. It will identify three questions that recur throughout the subject. Which value is recorded at a boundary? Which components meet there? What changes when the observation is refined?

2.1. **Reset values and sampled energy.** On $0 < x < 1$, take the line $f(x) = 2x - 1$ and repeat it with period one. The value approaches 1 before each integer and $-1$ after it. Assign the midpoint reading $f(0) = 0$, and the same reading at every integer. These assigned values are part of the field.

Sample at $0, 1/3, 2/3$. The curve and the three readings are shown below. Open circles mark the limiting values at the ends of the rising segment; filled circles mark actual samples.

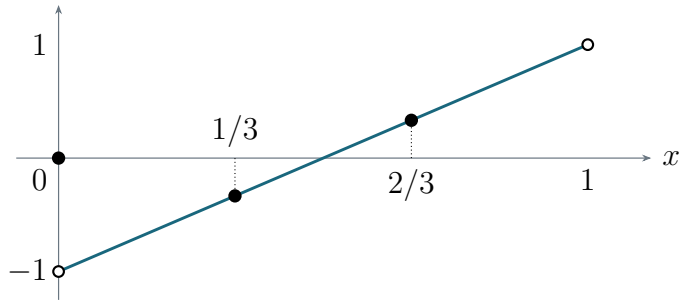



FIGURE 2. The ramp and its three samples. The periodic curve approaches 1 from the left and $-1$ from the right at zero. Its assigned reset reading is zero. Open circles mark traces; filled circles mark the observed readings.

The continuous energy and the sampled energy are

$$\int_0^1 (2x-1)^2\,dx = \frac{1}{3}, \qquad \frac{1}{3}\left(0^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = \frac{2}{27}.$$

Their difference, integral minus sample average, is $7/27$.

Now change only the reset reading to $g(0) = -1$. This is the right-hand reading of the same rising segments. Changing one point leaves the integral equal to $1/3$, but it increases the sample average by $1/3$. The two observations are

| Reset reading | Continuous energy | Sampled energy | Difference |
|---|---|---|---|
| Midpoint 0 | $1/3$ | $2/27$ | $7/27$ |
| Right-hand $-1$ | $1/3$ | $11/27$ | $-2/27$ |

The integral stays fixed while the reset reading changes the sampled energy (Figure 3).

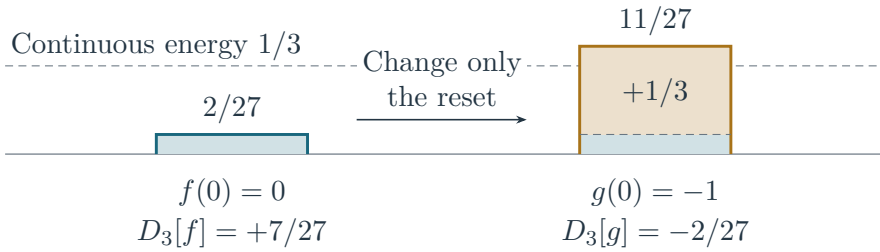



FIGURE 3. One reading reverses the comparison. The bars are sampled energies. Changing the value at zero adds 1/3 to the sampled energy and leaves the integral fixed. The gold part is exactly the additional sample contribution.

The same open curve has produced discrepancies of opposite signs. The integral and sample average are both correct for their definitions. The reading convention determines which comparison is being made.

2.2. **Point contribution and signed remainder.** At the reset, the two limiting energies are both one. Their average is one, whereas the midpoint reading has energy zero. A sampled point of weight 1/3 therefore contributes 1/3 to the discrepancy. The whole discrepancy is 7/27, leaving a signed contribution of $-2/27$ from the rest of the boundary formula.

The reset therefore explains more than the complete discrepancy. The remaining term is negative. With the right-hand reading, the point contribution vanishes and the same $-2/27$ is the whole answer. The boundary formula will locate that term in the change of slope of the squared field.

**Problem 1** (Changing one of the three readings)**.** Keep the open curve above and set its reset reading to a real number $c$. Find its sampled energy and discrepancy. For which $c$ is the discrepancy zero?

2.3. **Collision amplitudes and interaction.** Suppose a field has limiting values $u + v$ and $u - v$ on the two sides of a boundary, with the midpoint value $u$ recorded at the boundary. The average of the limiting squared magnitudes is $|u|^2 + |v|^2$. The recorded square is $|u|^2$. Their local difference is $|v|^2$.

If several components reset together, their signed jumps add before this square is taken. This coincidence of boundary events is a *collision*. Its contribution depends on the combined reading, including the interaction between the resetting components.

For example, add a second copy of the three-sample field $f$. The combined field is $2f$, whose discrepancy is $4(7/27) = 28/27$. Adding the two individual discrepancies gives only $14/27$. The missing $14/27$ is the mixed contribution. If the second field is $-f$ instead, the combined field and its discrepancy vanish. The mixed contribution then cancels the two individual discrepancies.

The square in Figure 4 includes the mixed area. Opposing amplitudes remove it along with the two individual squares.

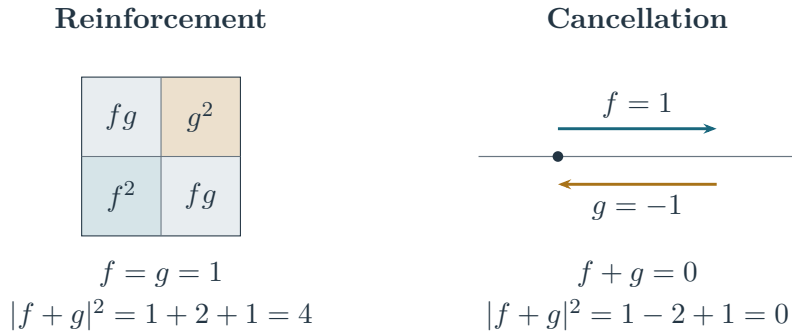



FIGURE 4. At one address, the mixed contribution decides the combined energy. The two component squares are one in both cases. On the left the two cross rectangles add two; on the right the signed mixed term subtracts two.

Both calculations are consequences of

$$|f + g|^2 = |f|^2 + 2\,\mathrm{Re}(f\overline{g}) + |g|^2.$$

The mixed term must be observed using the same integral or sampling rule as the two squares, including when that rule changes.

2.4. **Grid refinement and field growth.** For the same midpoint field $f$, now sample at $a/q$, $0 \le a < q$. Elementary sums of integers

and their squares give

$$\frac{1}{q}\sum_{a=1}^{q-1}(2a/q-1)^2 = \frac{1}{3}-\frac{1}{q}+\frac{2}{3q^2}.$$

The discrepancy is therefore

$$D_q[f] = \frac{1}{q}-\frac{2}{3q^2} \longrightarrow 0. \tag{1}$$

Here $D_q$ is simply a name for integral minus sampled mean square. The field was held fixed while the grid changed.

Now let the field grow with that grid by taking $f_q = \sqrt{q}\, f$. Both observed energies are multiplied by $q$, so the complete discrepancy is

$$D_q[f_q] = 1-\frac{2}{3q} \longrightarrow 1. \tag{2}$$

This limit has been calculated directly, including its signed remainder. There is no conflict with the fixed-field limit. The changing amplitude has exactly offset the decreasing weight of one sample.

In Figure 5, the shrinking point mass wins for the fixed field. The growing amplitude compensates for it in the second family.

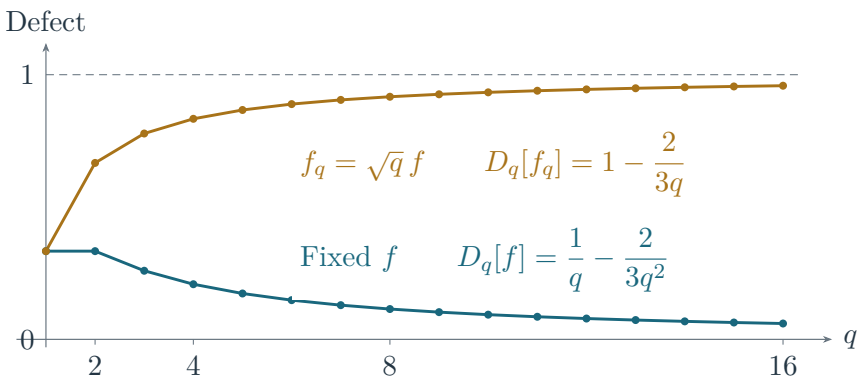



FIGURE 5. The same grids give different limits when the field grows. Dots show integer grid sizes; the connecting segments guide the eye. Multiplying the field by $\sqrt{q}$ multiplies both energies and their difference by $q$. The gold sequence approaches one, while the blue sequence approaches zero.

A collision of $m$ equal components has squared amplitude proportional to $m^2$. On a $q$-point grid its point contribution can consequently have size $m^2/q$. This explains why the scale $m \asymp \sqrt{q}$ deserves attention. In a general growing family the remaining signed terms still require an estimate; formula (2) works because those terms were also computed.

**Problem 2** (Choose the growth scale)**.** For real $\alpha$, put $f_q = q^\alpha f$. Determine when the discrepancy tends to zero, tends to one, or grows without bound.

2.5. **Unity and refinement.** The changing ramp raises a separate question about what refinement preserves. The constant field gives a reference against which to distinguish subdivision of the observation from changes in its readings.

Unity is the constant field $\mathbf{1}(x) = 1$. It gives a fixed reference while a description becomes more detailed. Both observations in the opening example assign total mass one: the interval has length one, and the three sample weights add to one. Consequently both assign energy one to unity. Their disagreement about the sawtooth comes from its readings at those same addresses.

A whole can be resolved into unequal shares. The reciprocal weights of the later native arithmetic observation are

$$w_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \sum_{n=1}^{M} w_n = 1 - \frac{1}{M+1}.$$

The last term tends to zero, so all positive addresses together have mass one. Resolving the first two addresses leaves

$$1 = \underbrace{\frac{1}{2}}_{n=1} + \underbrace{\frac{1}{6}}_{n=2} + \underbrace{\frac{1}{3}}_{n\geq 3}.$$

Resolving the next address divides the last share into $1/12 + 1/4$. The remaining $1/4$ still includes infinitely many addresses. It is part of the same total at every stage.

Figure 6 shows the first two such resolutions.

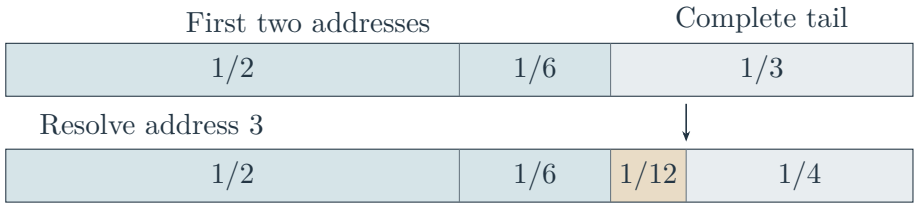



FIGURE 6. Refinement divides the remaining mass. Bar lengths are the actual weights. The tail 1/3 becomes the newly resolved share 1/12 and the complete remaining tail 1/4. Both observations give energy one to the unchanged constant field.

This is *conservation of mass under subdivision.* If a cell of mass $m$ is partitioned into children of masses $m_1, \ldots, m_k$, then

$$\sum_{j=1}^{k} m_j = m, \qquad \sum_{j=1}^{k} m_j |c|^2 = m|c|^2.$$

The second identity follows when each child inherits the same reading $c$. Finer resolution alone has preserved that field's energy. New child readings require a new energy calculation. The observation mass, the constant field unity and the energy of a general field are distinct objects. On an observation of mass one, the constant field two has energy four.

A resolved part and its complement. Let $u_M(n)$ be one on addresses $1, \ldots, M$ and zero on all later addresses. This indicator field represents unity exactly on the resolved part. Its residual $r_M = \mathbf{1} - u_M$ is one on the remaining tail. Their supports are disjoint, so

$$\sum_{n \geq 1} w_n |u_M(n)|^2 = 1 - \frac{1}{M+1}, \qquad \sum_{n \geq 1} w_n |r_M(n)|^2 = \frac{1}{M+1}.$$

For $M = 2$, the represented energy is 2/3 and the error energy is 1/3. Admitting the third indicator transfers 1/12 from the error to the represented part. The target remains $\mathbf{1}$ throughout. Here the allowed fields are indicators; a prescribed clock family offers different corrections.

This fixed-target account motivates the refinement work in *Unity, Refinement, and Signed Capture* [22]. Section 10.1 develops its elementary clock example. Corrections there can overlap the old fit, so their signed interaction enters the error change. The mass partition identifies the whole being observed; the permitted fields determine how much of unity can be represented.

**Problem 3** (Unity and unequal shares)**.** Split an observation of mass one into two addresses with masses $1/3, 2/3$. Give them real readings $1 + a, 1 + b$. Choose $b$ in terms of $a$ so that the weighted mean remains one. Compute the field's energy and its squared error relative to unity, including the case $a = 2$. Compare with the choice $b = -a$. Explain which quantities subdivision preserves and which depend on the child readings.

2.6. **Interval pulse.** Take a unit pulse supported on $[1/4, 3/4)$ and sample at $0, 1/4, 1/2, 3/4$. The continuous energy is the interval length $1/2$. Two of the four readings are one, so the sampled energy is also $1/2$. The discrepancy is zero.

Change the pulse to have value zero at both endpoints. The integral stays $1/2$, but only the sample at $1/2$ remains. The sampled energy is now $1/4$ and the discrepancy is $1/4$. This calculation uses the same point-replacement rule as the clock example. Rising segments and arithmetic divisibility are not required for that operation.

The later pulse calculation allows any interval, amplitude, endpoint readings and grid offset. It will show how a boundary formula replaces a fresh case-by-case count each time the sampling grid moves.

2.7. **Changes of field and observation.** The starting question asks how an observed squared norm changes when its field or observation changes. The data needed for an answer depend on that change.

| Change | Data that must be retained |
|---|---|
| Change a reading | The actual value and the weight of that sample. |
| Add a field | Its square and its interaction with the inherited field. |
| Move or refine a grid | Which boundaries are sampled and the new offset. |
| Grow the field | The dependence of every error constant on that field. |
| Change weights or activation | Both endpoints, weight variation and the complete tail. |

Each change can alter the comparison between an integral and a finite squared observation. The examples will cover periodic signals, interval indicators, arithmetic carry fields and reciprocal address weights. Each application specifies its observation before applying the common rules.

For the next problem, a grid with $q$ points and offset $\theta$ samples $(a+\theta)/q$, $0 \le a < q$. The offset is measured in cells: $\theta = 1/2$ moves each sample by half a cell, or $1/(2q)$.

**Problem 4** (An isolated reading)**.** Let $u$ vanish on the periodic interval except for $u(0) = 2$. Compute its continuous energy and its sampled energy on the four-point grids with offsets $\theta = 0$ and $\theta = 1/2$. Determine both defects. Explain which datum makes the answers different.

## 3. Fields and observations

A field supplies readings. An observation assigns mass to the addresses at which those readings are taken. Here an address is simply a location in the domain, such as a point of an interval or an integer index. Keeping both objects visible lets the same calculation describe a grid, an interval integral, a moving window or an infinite weighted tail.

A two-address example separates the two objects. Give the readings $1, 3$ masses $1/2, 1/2$. The observed energy is $(1^2 + 3^2)/2 = 5$. Keep the readings but change the masses to $3/4, 1/4$; the energy becomes 3. Keep those new masses and change the second reading to 1;

the energy becomes 1. The first change belongs to the observation, the second to the field. We need notation that records both.

A stem in Figure 7 shows the reading. Its mass determines how much the squared reading contributes.

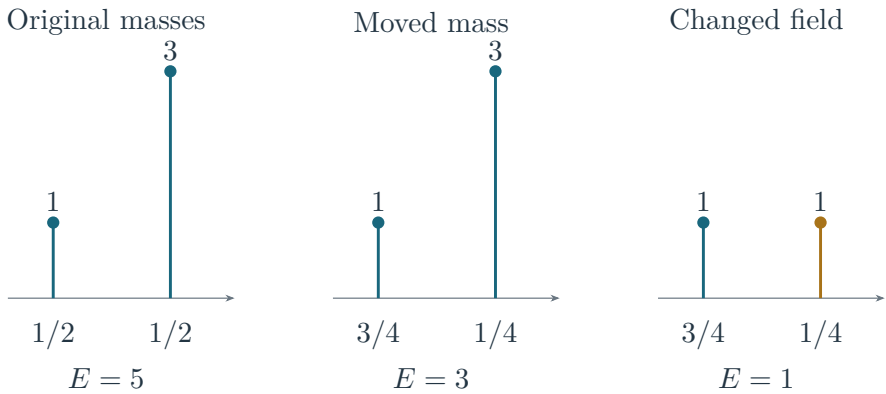



FIGURE 7. Changing a mass and changing a reading are separate operations. The first two panels observe the same field $(1, 3)$. The third retains the new masses and changes the field to $(1, 1)$. Stem heights are readings; the fractions at their feet are observation masses. The energies below use the weighted squares.

For a finite table, the definitions below reduce to these weighted sums. A measure records the mass assigned to each measurable set of addresses. Its integral adds readings with those masses, extending the same operation to a continuous profile or an infinite weighted tail.

Let $X$ be a measurable address space whose singletons are measurable. An observation $\rho$ is a finite positive measure on $X$; its mass need not be one. A field is a specified measurable function on $X$, including its readings at atoms of $\rho$. Define

$$Q_\rho h = \int_X h \, d\rho, \qquad E_\rho[f] = Q_\rho(|f|^2), \qquad K_\rho(f, g) = Q_\rho(f\overline{g}).$$

We use $Q_\rho$ only when the integral exists and use $E_\rho$ and $K_\rho$ on fields of finite energy. Cauchy–Schwarz then gives $|K_\rho(f, g)|^2 \le E_\rho[f]E_\rho[g]$. The letters name the operation and the subscripts identify its observation. These are the usual integral, squared norm and

mixed form for the chosen measure. Keeping the measure visible prevents a change of addresses or masses from slipping unnoticed into the calculation.

When two observations $\rho, \sigma$ are compared, write

$$D_{\rho,\sigma}[f] = E_\rho[f] - E_\sigma[f], \qquad K_{\rho,\sigma}(f,g) = K_\rho(f,g) - K_\sigma(f,g).$$

Both fields must have finite energy in both observations. The difference $D$ can have either sign. It is quadratic in the field, and its mixed form $K_{\rho,\sigma}$ need not be positive.

| Observation | Measure on its address space |
|---|---|
| Periodic integral | $dx$ on $\mathbb{T}$. |
| Shifted grid | $q^{-1}\sum_{a=0}^{q-1}\delta_{(a+\theta)/q}$. |
| Weighted window | $\sum_{a<n\le b} v_n\delta_n$, with $v_n \ge 0$. |
| Native tail | $\sum_{n>S}[n(n+1)]^{-1}\delta_n$. |

Here $\delta_x$ assigns mass one to the single address $x$. Thus an entry $v\delta_x$ in the observation means "read at $x$ with weight $v$." A collection of such point masses turns the integral into the corresponding weighted sum.

In the grid setting the first comparison measure is understood to be $dx$, so $D_{q,\theta}$ and $K_{q,\theta}$ abbreviate these two-measure operations. Later $E_S$ abbreviates the native tail. Normalizing a tail to mass one would define a different observation and change its energy.

The midpoint ramp from Section 2 now has a compact description. Its field stays fixed while the measures $dx$ and $\frac{1}{3}(\delta_0+\delta_{1/3}+\delta_{2/3})$ give energies $1/3$ and $2/27$. Replacing its reading at zero changes the field instead. The notation below keeps these two operations separate in one identity.

3.1. **Energy under changes of field and observation.** Suppose the old experiment observes $f$ with $\rho$, and the new one observes $f+g$ with $\sigma$. The field $f$ must be specified at every new address, even where the old observation assigned no mass.

**Proposition 3.1** (Change of field and observation)**.** *If $f$ has finite energy under $\rho$ and $\sigma$, and $g$ has finite energy under $\sigma$, then*

$$E_\sigma[f+g] - E_\rho[f] = -D_{\rho,\sigma}[f] + 2\,\mathrm{Re}\,K_\sigma(f,g) + E_\sigma[g]. \tag{3}$$

*For a common atomic address space with old masses $u_n$ and new masses $v_n$, the first term is $\sum_n (v_n - u_n)|f(n)|^2$. It includes every removed address, newly admitted address and change of retained mass.*

*Proof.* Expand $|f+g|^2 = |f|^2 + 2\,\mathrm{Re}(f\overline{g}) + |g|^2$ under $\sigma$ and subtract $E_\rho[f]$. The two square integrals converge by hypothesis, and Cauchy–Schwarz makes the mixed integral absolutely convergent. In the atomic case $\sum_n |v_n - u_n||f(n)|^2 \le E_\sigma[f] + E_\rho[f]$, so its series also converges absolutely. This proves the formula without an unjustified rearrangement of an infinite sum. □

The three terms answer different parts of the calculation. The first charges the old field for changing its observation. The second measures the added field's signed interaction with what was already present. The third is the added field's own energy.

For a small calculation, give addresses $1, 2, 3$ old masses $(1/2, 1/3, 1/6)$ and new masses $(0, 1/2, 1/2)$. Take $f = (1, 2, 0)$ and $g = (0, -1, 1)$. The energies are $11/6$ and $1$. The three terms on the right are respectively $1/6, -2, 1$. Their sum is $-5/6$. Removing address one does not determine the answer by itself; reweighting address two and its negative mixed term also contribute.

3.2. **Readings, superposition and consistency.** If two fields differ only on an at most countable set $A$, have finite energy, and agree elsewhere, then

$$E_\rho[g] - E_\rho[f] = \sum_{a \in A} \rho(\{a\})\big(|g(a)|^2 - |f(a)|^2\big).$$

This is the general point-value replacement rule. Its series is absolutely convergent. An atomless observation ignores the changed readings; an atomic observation charges their actual masses.

For finitely many components $f = \sum_j c_j f_j$, the matrix $G_{ij} = K_\rho(f_j, f_i)$ gives $E_\rho[f] = c^* G c$. Here a collision means that components contribute at a common observed address; the mixed term measures their interaction. Components with disjoint observed support have zero mixed energy. Components meeting at a boundary contribute through their combined reading. The interaction may reinforce or cancel in the combined energy.

Changing an experiment in two stages gives a useful consistency check. Apply (3) to each stage and add. The intermediate energy cancels, leaving the direct change. The allocation among the three terms can depend on the chosen intermediate field and observation; their total cannot. This check catches a lost shell or a duplicated boundary credit before any estimate is attempted.

**Problem 5** (Changing masses and readings)**.** On three addresses take old masses $(1/2, 1/2, 0)$, new masses $(0, 1/3, 2/3)$, old field $f = (1, -1, 2)$ and change $g = (0, 1, -1)$. Calculate the energy change directly. Then recover it as the sum of observation change, inherited interaction and new square. Keep the old field's reading at the newly admitted address.

An interval integral and a weighted grid are both observations in this sense. For that comparison, a field's boundary readings need more detail than a finite vector supplies. The next section specifies the traces and jumps that the boundary formula will use.

## 4. Fields, traces and sampled energy

The first example distinguished a curve from its assigned readings. We now record that distinction for any periodic field with finitely many boundaries. Work on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the unit interval with its ends identified. Write $\lfloor x \rfloor$ for the greatest integer at most $x$, $\lceil x \rceil$ for the least integer at least $x$, and $\{x\} = x - \lfloor x \rfloor$ for the fractional part. The notation $\mathbf{1}_A$ is one on the set $A$ and zero elsewhere.

An admissible field $f : \mathbb{T} \to \mathbb{C}$ has a finite set $\mathcal{B}$ of specified boundary points. On each open arc between consecutive points of $\mathcal{B}$, it extends to a $C^2$ function on the closed arc. The extensions from adjacent arcs may differ. The actual value $f(b)$ is specified separately at every $b \in \mathcal{B}$. Include isolated exceptional readings in $\mathcal{B}$, even if the two traces agree. A smooth field may be given an artificial boundary at zero.

The $C^2$ condition means that the function and its first two derivatives are continuous on each closed piece. It permits integration by parts inside each piece while keeping its endpoint terms. A trace is the limiting value reached from one side. In the opening sawtooth, the left trace at zero is 1, the right trace is $-1$, and the assigned reading is 0. These are three distinct pieces of data.

For any function $h$ with these properties, write

$$[h]_b = h(b+) - h(b-), \qquad \langle h \rangle_b = \frac{h(b+) + h(b-)}{2},$$
$$a_b(h) = \langle h \rangle_b - h(b).$$

All traces at zero are periodic traces. The notation $h''_{\mathrm{reg}}$ means the ordinary second derivative on the open arcs, ignoring the finitely many boundary points. The point data are not identified modulo sets of measure zero.

For an integer $q \geq 1$ and $0 \leq \theta < 1$, define

$$X_{q,\theta} = \left\{ \frac{a+\theta}{q} : 0 \leq a < q \right\}, \qquad Q_{q,\theta} h = \frac{1}{q} \sum_{a=0}^{q-1} h\left( \frac{a+\theta}{q} \right).$$

The linear sampling discrepancy and the quadratic energy defect are

$$L_{q,\theta}(h) = \int_0^1 h \, dx - Q_{q,\theta} h, \qquad D_{q,\theta}[f] = L_{q,\theta}(|f|^2).$$

We abbreviate $Q_{q,0}$ and $D_{q,0}$ to $Q_q$ and $D_q$. The square is taken before sampling. Define the periodic kernels

$$\psi(t) = \begin{cases} \{t\} - 1/2, & t \notin \mathbb{Z}, \\ 0, & t \in \mathbb{Z}, \end{cases} \qquad \widetilde{B}_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}.$$

The first kernel takes its midpoint value at a jump. The second is continuous. Their respective absolute bounds are $1/2$ and $1/6$. Ordinary letters $Q, D, K$ denote sampling, energy defect and mixed defect. The offset may be suppressed only after its value has been fixed. The kernels will record where a boundary falls within a sample cell. For a boundary $b$, the number $qb - \theta$ is an integer precisely when the boundary is sampled. Its fractional part records the relative position otherwise. Thus the apparently global sampling problem has local phase data at each boundary.

The signs of the phase kernels in Figure 8 determine how each boundary contributes.

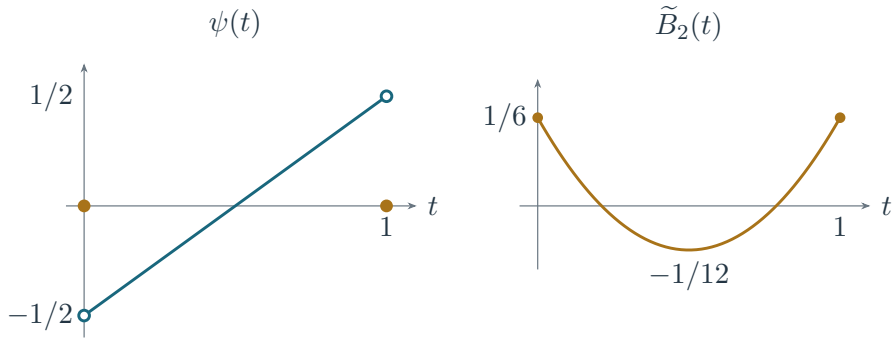



FIGURE 8. The two periodic kernels read a boundary's phase within a grid cell. The first has the assigned value zero at each integer. The second joins continuously across the reset. Their vertical scales differ; both have mean zero over a full period.

For the opening field $f = 2\psi$, form its energy profile first. At zero, $h = f^2$ has reading zero and both traces one. Thus $a_0(h) = 1$ and $[h]_0 = 0$. Its slope traces are 4 and $-4$, so $[h']_0 = -8$, while $h''_{\mathrm{reg}} = 8$. These four entries will reproduce $D_q[f] = 1/q - 2/(3q^2)$ in the next section.

Squaring makes the two energy traces agree, but leaves the assigned energy at zero (Figure 9).

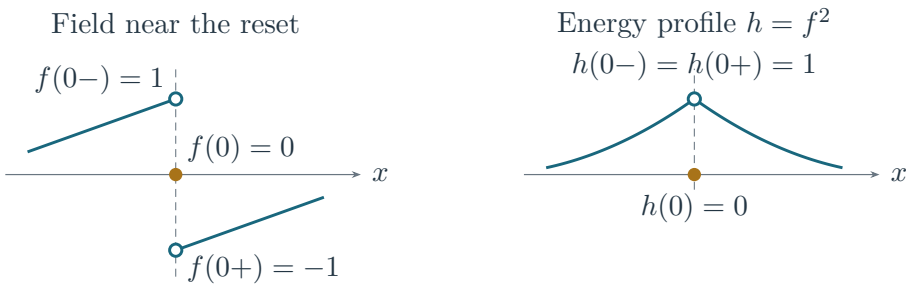



FIGURE 9. Squaring removes the value jump of the opening ramp but leaves its exceptional reading. The two energy traces agree at one; the assigned energy is zero. Thus $[h]_0 = 0$ and $a_0(h) = 1$. The horizontal scale is enlarged near the periodic reset.

4.1. **Boundary data.** The periodic integral identifies functions that differ at finitely many points. A finite sampling rule need not do so. For example, let $f$ vanish everywhere except at zero, where $f(0) = a$. Then

$$\int_0^1 |f|^2 \, dx = 0, \qquad D_q[f] = -\frac{|a|^2}{q}.$$

Both traces at zero vanish. The entire defect comes from the specified point value. This field is admissible after zero is included in $\mathcal{B}$. If the grid misses zero, its defect is zero as well.

The two measures being compared are Lebesgue measure and the atomic measure $q^{-1} \sum_{x \in X_{q,\theta}} \delta_x$. Their difference is a signed measure. A positive defect means the continuous energy exceeds the sampled energy; a negative defect means the sampled energy exceeds the continuous energy. Neither sign is prescribed in advance.

| Symbol | Data or operation |
|---|---|
| $q, \theta$ | Number of samples and offset measured in grid cells. |
| $\mathcal{B}$ | Finite set containing all jumps and exceptional readings. |
| $[h]_b$ | Right trace minus left trace at a boundary. |
| $a_b(h)$ | Average trace minus the actual point reading. |
| $L_{q,\theta}$ | Linear difference of integral and sample average. |
| $D_{q,\theta}$ | The same difference applied to squared magnitude. |
| $K_{q,\theta}$ | Mixed energy difference for two fields. |
| $P_{q,\theta}, R_{q,\theta}$ | Sampled point contribution and the remaining signed defect. |
| $E_S$ | Native weighted energy on addresses strictly greater than $S$. |
| $N$ | Local size parameter: a cutoff in weighted sums, a scale in the moving-window model, and the clock count $p - 1$ in the carry application. |

4.2. **Point-value replacement.**

**Proposition 4.1** (Point-value replacement)**.** *Suppose admissible fields $f, g$ agree outside a finite set $S$. Then*

$$D_{q,\theta}[g] - D_{q,\theta}[f] = \frac{1}{q} \sum_{b \in S \cap X_{q,\theta}} \left( |f(b)|^2 - |g(b)|^2 \right).$$

*Proof.* The integrals of squared magnitude agree. In the sample sums, only the readings in $S \cap X_{q,\theta}$ can differ. Subtracting those sums gives the result. □

This identity permits a calculation to use a convenient trace convention and then restore the required readings exactly. The correction depends on which points the grid samples. Changing an unsampled point has no effect.

**Problem 6** (Boundary readings and traces)**.** On the periodic interval let $f = 1$ on $(0, 1/2)$ and $f = 3$ on $(1/2, 1)$, with $f(0) = f(1/2) = 2$. Record the two traces and the actual reading of $f$ and $h = f^2$ at both boundaries. Find $a_b(h)$ and $[h]_b$. Compute $Q_q(h)$, $D_q[f]$ and the sampled point contribution for $q = 3, 4$. Compare $Q_q(f^2)$ with $(Q_q f)^2$.

## 5. Boundary decomposition

The three-sample calculation had a point contribution and a signed remainder. The following formula separates that remainder into value jumps, slope jumps and curvature between boundaries. Its four terms can be calculated from local data before any asymptotic estimate is made.

Euler–Maclaurin summation supplies the underlying relation between sums, integrals and boundary data. Its circular formulation treats interior and periodic jumps [3]; shifted sampling with jumps is developed by Berrut and Trummer [4]. The Bernoulli kernels used below are defined in Section 4; their Fourier expansions are recorded in [10, Section 24.8]. Stein and Shakarchi [26] develop the Fourier-series methods used in the proof. For energy calculations, the boundary data belong to the squared field. They include its assigned readings and all interactions between components.

The formula needs four kinds of data. At each boundary record the actual energy and its two traces, then the two slope traces. Between boundaries retain the ordinary second derivative. The grid supplies the phase $qb - \theta$ at each boundary. The theorem weights those entries by two fixed periodic functions. Its first term records only boundaries actually sampled; the remaining terms account for the surrounding profile. The proof derives those weights from Fourier series, but the calculation itself uses the listed local data.

**Theorem 5.1** (Boundary decomposition)**.** *Let $h : \mathbb{T} \to \mathbb{C}$ be piecewise $C^2$ with specified point values as above. For every $q \geq 1$ and $0 \leq \theta < 1$,*

$$L_{q,\theta}(h) = \frac{1}{q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} a_b(h) - \frac{1}{q} \sum_{b \in \mathcal{B}} [h]_b \psi(qb - \theta) \tag{4}$$
$$+ \frac{1}{2q^2} \sum_{b \in \mathcal{B}} [h']_b \widetilde{B}_2(qb - \theta) + \frac{1}{2q^2} \int_0^1 h''_{\mathrm{reg}}(x) \widetilde{B}_2(qx - \theta)\, dx.$$

*In particular, this formula with $h = |f|^2$ gives the complete energy defect of every admissible field. Boundary coincidences with the grid are included without a limiting convention for the sampled value.*

Figure 10 separates the four contributions. Only the first is confined to boundaries that the grid actually samples.

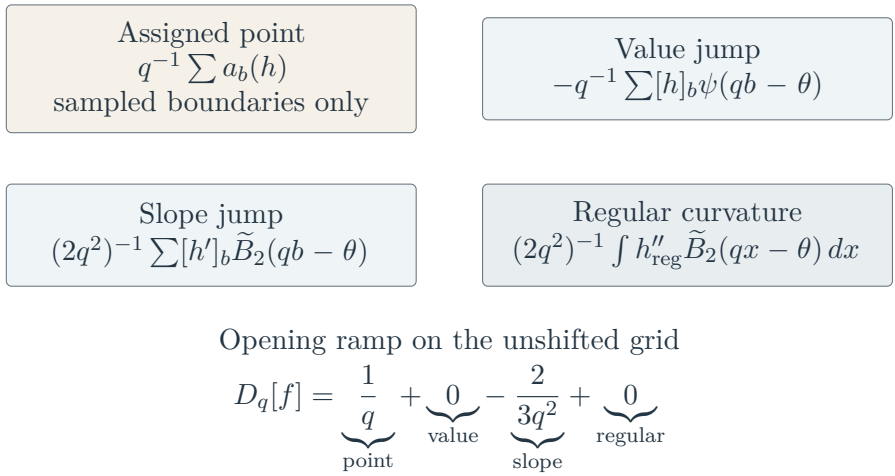



FIGURE 10. Four contributions make the complete boundary formula. Every box acts on the energy profile $h = |f|^2$. The opening ramp uses only the point and slope-jump terms. For another field, the other two boxes may carry part or all of the defect.

5.1. **The opening ramp.** For $f(x) = 2x - 1$ on $0 < x < 1$ with $f(0) = 0$, the energy data listed in Section 4 are

$$a_0(h) = 1, \qquad [h]_0 = 0, \qquad [h']_0 = -8, \qquad h''_{\mathrm{reg}} = 8.$$

The unshifted grid samples zero. Its point contribution is $1/q$. The value-jump term vanishes, and the derivative term is $(-8)/(12q^2) = -2/(3q^2)$. The regular curvature is constant and the Bernoulli kernel has mean zero, so their integral vanishes. The formula gives the complete answer

$$D_q[f] = \frac{1}{q} - \frac{2}{3q^2}.$$

At $q = 3$ this is $7/27$. Changing the reset reading to $-1$ removes the point contribution and leaves $-2/27$. The derivative jump accounts for the signed remainder that the opening calculation left unexplained.

5.2. **Derivation of the boundary formula.** The proof uses a different description of the same data. Integration by parts records the jumps in each Fourier coefficient. Averaging a Fourier mode over the grid removes it unless its frequency is a multiple of $q$. Summing

the surviving modes produces the two Bernoulli kernels. The point replacement at the end restores the actual boundary readings.

*Proof.* First replace $h(b)$ by $\langle h\rangle_b$ at each boundary, and call the result $h_*$. The integral does not change, and

$$L_{q,\theta}(h) = L_{q,\theta}(h_*) + \frac{1}{q}\sum_{b\in\mathcal{B}\cap X_{q,\theta}} a_b(h). \tag{5}$$

Put $\widehat{h}(n) = \int_0^1 h(x)e^{-2\pi inx}\,dx$. Integration by parts twice on every arc, with the periodic endpoint terms retained, gives for $n \neq 0$

$$(2\pi in)^2\widehat{h}(n) = \widehat{h''_{\mathrm{reg}}}(n) + \sum_b [h']_b e^{-2\pi inb} + 2\pi in\sum_b [h]_b e^{-2\pi inb}. \tag{6}$$

Equivalently, the distributional second derivative has the regular part $h''_{\mathrm{reg}}$ and the point terms $\sum_b [h']_b\delta_b + \sum_b [h]_b\delta'_b$.

For $0 < \rho < 1$, the Abel means $h_\rho(x) = \sum_n \rho^{|n|}\widehat{h}(n)e^{2\pi inx}$ converge to $h_*(x)$ at every point. Indeed, the Poisson kernel is even and has total mass one; its mass outside any neighborhood of zero tends to zero. Its two half-neighborhoods consequently select the average of the two traces. Sampling the absolutely convergent series and using finite geometric sums yields

$$Q_{q,\theta}h_\rho - \int_0^1 h\,dx = \sum_{\ell\neq 0} \rho^{q|\ell|}\widehat{h}(q\ell)e^{2\pi i\ell\theta}.$$

Insert (6). The kernel identities are

$$\lim_{r\uparrow 1}\sum_{\ell\neq 0}\frac{r^{|\ell|}e^{-2\pi i\ell t}}{2\pi i\ell} = \psi(t),$$

$$\sum_{\ell\neq 0}\frac{e^{-2\pi i\ell t}}{(2\pi i\ell)^2} = -\frac{1}{2}\widetilde{B}_2(t).$$

They follow from the Fourier coefficients of the two Bernoulli functions; the first has value zero at integers by pairing opposite frequencies. The second series converges absolutely and uniformly. The regular integral may therefore be interchanged with its series and limit, since $h''_{\mathrm{reg}}$ is integrable. The boundary sums are finite. Taking $\rho\uparrow 1$ proves (4) for $h_*$; (5) restores the prescribed readings. □

5.3. **Point corrections and residual bounds.**

**Corollary 5.2** (Midpoint reading)**.** *If* $f(b) = (f(b+) + f(b-))/2$, *then*

$$a_b(|f|^2) = \frac{1}{4}|[f]_b|^2.$$

*Proof.* Write $u = (f(b+) + f(b-))/2$ and $v = (f(b-) - f(b+))/2$. The identity $(|u+v|^2 + |u-v|^2)/2 = |u|^2 + |v|^2$ gives the claim. □

The midpoint of a field and the midpoint of its energy are different readings. At an integer-zero sawtooth reset the field takes its midpoint, so its energy is smaller than the midpoint of the two energy traces. The resulting point correction is positive. The signed terms in (4) need not be positive.

**Corollary 5.3** (A sufficient residual estimate)**.** *Set*

$$A_{q,\theta}(h) = \frac{1}{q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} a_b(h).$$

*For every constant* $c \in \mathbb{C}$,

$$|L_{q,\theta}(h) - A_{q,\theta}(h)| \le \frac{1}{2q} \sum_b |[h]_b| + \frac{1}{12q^2} \left( \sum_b |[h']_b| + \int_0^1 |h''_{\mathrm{reg}}(x) - c|\, dx \right).$$

*For a sequence* $h_\nu, q_\nu, \theta_\nu$, *if the right side tends to zero and* $A_{q_\nu,\theta_\nu}(h_\nu) \to A$, *then* $L_{q_\nu,\theta_\nu}(h_\nu) \to A$.

*Proof.* The integral of $\widetilde{B}_2(qx - \theta)$ over $\mathbb{T}$ is zero. Subtract $c$ in the regular integral and use the two kernel bounds. □

This is an absolute sufficient condition. A growing arithmetic family may require cancellation in the signed sums when their absolute bounds do not decay. The decomposition identifies those sums without supplying an estimate for them.

5.4. **Polynomial curvature.** The ramp's constant regular curvature made its integral term vanish. A curved field can retain a further contribution. The next calculation evaluates it, so every term of the formula has a worked example.

Let $f(x) = x^2$ on $0 \le x < 1$ and extend it periodically, retaining $f(0) = 0$. Its energy profile is $h(x) = x^4$ on that interval. At the periodic join, the boundary data are

$$a_0(h) = \frac{1}{2}, \qquad [h]_0 = -1, \qquad [h']_0 = -4, \qquad h''_{\mathrm{reg}}(x) = 12x^2.$$

On the unshifted $q$-grid the value-jump term vanishes because $\psi(0) = 0$. The point and derivative terms are $1/(2q)$ and $-1/(3q^2)$. To calculate the regular term, split the integral into the $q$ cells and set $qx = a + t$ in cell $a$. Since

$$\int_0^1 \widetilde{B}_2(t)\,dt = \int_0^1 t\widetilde{B}_2(t)\,dt = 0, \qquad \int_0^1 t^2\widetilde{B}_2(t)\,dt = \frac{1}{180},$$

direct expansion gives

$$\int_0^1 x^2\widetilde{B}_2(qx)\,dx = \frac{1}{q^3}\sum_{a=0}^{q-1}\int_0^1 (a+t)^2\widetilde{B}_2(t)\,dt = \frac{1}{180q^2}.$$

Thus the complete answer is

$$D_q[f] = \frac{1}{2q} - \frac{1}{3q^2} + \frac{1}{30q^4}.$$

For $q = 2$ it is $27/160$. Directly, the integral of the squared field is $1/5$ and the two sampled squares have mean $1/32$, giving the same result. The $q^{-4}$ contribution would disappear if regular curvature were discarded. The point and jump terms alone would miss that part of the finite answer.

**Problem 7** (A periodic ramp)**.** Let $f(x) = x$ on $0 \le x < 1$ and repeat it periodically, with $f(0) = 0$. Find $D_q[f]$ for every $q \ge 1$ in two ways. First use a sum of squares; then list the point correction, value jump, derivative jump and regular curvature of $|f|^2$. Explain why the regular integral vanishes even though that curvature is nonzero.

**Problem 8** (Smooth resonance)**.** For an integer $m \geq 1$, take $f_m(x) = \sin(2\pi m x)$. Compute its continuous energy and its sampled energies on the unshifted grids $q = 2m$ and $q = 2m + 1$. Identify the term of the boundary formula that accounts for any nonzero defect. For fixed $m$, determine the defect whenever $q > 2m$. Explain the distinction between that statement and the sequence of fields sampled at $q = 2m$.

## 6. Simultaneous arithmetic resets

For clocks, the local data have a particularly simple form. Each reset has a known jump, and each open segment has a constant slope. Simultaneous resets must first be grouped into a single signed amplitude. The clock $2\psi(2x)$ repeats the opening ramp twice during one trip around the circle, resetting at zero and one half. In a general clock $2c\psi(rx + \phi)$, $r$ counts repetitions, $\phi$ chooses the phase, and $c$ sets the signed or complex amplitude. The three parameters change different parts of the boundary calculation.

Let $r_1, \dots, r_s$ be positive integers, let $\phi_1, \dots, \phi_s \in \mathbb{T}$, and let $c_1, \dots, c_s \in \mathbb{C}$. Set

$$F(x) = 2\sum_{j=1}^{s} c_j \psi(r_j x + \phi_j), \qquad V = 2\sum_{j=1}^{s} c_j r_j.$$

Let $\mathcal{B}$ be the union of all reset locations. At $b \in \mathcal{B}$ define the collision set and its amplitude by

$$I_b = \{j : r_j b + \phi_j \in \mathbb{Z}\}, \qquad C_b = \sum_{j \in I_b} c_j.$$

The nonreset reading is $U_b = 2\sum_{j \notin I_b} c_j \psi(r_j b + \phi_j)$. Every location is counted once, with all its resetting clocks included in $I_b$.

**Theorem 6.1** (Clock collision formula)**.** *For all such clocks, every integer $q \geq 1$ and every $0 \leq \theta < 1$,*

$$D_{q,\theta}[F] = \frac{1}{q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} |C_b|^2 + \frac{4}{q} \sum_{b \in \mathcal{B}} \mathrm{Re}(U_b \overline{C_b}) \psi(qb - \theta) \qquad (7)$$
$$- \frac{2}{q^2} \sum_{b \in \mathcal{B}} \mathrm{Re}(V \overline{C_b}) \widetilde{B}_2(qb - \theta).$$

*No coprimality or distinct-frequency hypothesis is required.*

*Proof.* At each boundary,

$$F(b-) = U_b + C_b, \quad F(b+) = U_b - C_b, \quad F(b) = U_b.$$

For $h = |F|^2$, these give

$$a_b(h) = |C_b|^2, \qquad [h]_b = -4\,\mathrm{Re}(U_b\overline{C_b}), \qquad [h']_b = -4\,\mathrm{Re}(V\overline{C_b}).$$

On every open arc $F' = V$ and $F'' = 0$, so $h''_{\mathrm{reg}} = 2|V|^2$ is the same constant on every arc. Its Bernoulli integral vanishes. Theorem 5.1 now gives (7). □

For unit coefficients, a collision of $m$ clocks contributes $m^2/q$ if the grid samples it. With general coefficients,

$$|C_b|^2 = \sum_{j\in I_b} |c_j|^2 + 2\,\mathrm{Re} \sum_{\substack{j<k \\ j,k\in I_b}} c_j\overline{c_k}.$$

The reset budget already contains interference. Two clocks with the same frequency and phase and coefficients $1, -1$ have zero total field and zero collision amplitude at every reset. A count of individual resets alone would miss this cancellation.

6.1. **Two resetting clocks.** For $F = 2\psi(x) + 2\psi(2x)$, the slope on every open segment is $V = 6$. The distinct boundary locations are zero and $1/2$. Their data are

| $b$ | $C_b$ | $U_b$ | $V$ |
|---|---|---|---|
| 0 | 2 | 0 | 6 |
| 1/2 | 1 | 0 | 6 |

Thus the interaction term vanishes at both locations. On the unshifted nine-point grid, only zero is sampled, and

$$\sum_b C_b\widetilde{B}_2(9b) = 2\widetilde{B}_2(0) + \widetilde{B}_2(1/2) = \frac{1}{4}.$$

The clock formula gives

$$D_9[F] = \frac{4}{9} - \frac{12}{81}\frac{1}{4} = \frac{11}{27}.$$

On the half-cell grid, only $1/2$ is sampled. Now

$$\sum_b C_b\widetilde{B}_2(9b - 1/2) = 2\widetilde{B}_2(1/2) + \widetilde{B}_2(0) = 0,$$

and $D_{9,1/2}[F] = 1/9$. The continuous field has not changed. The two sampling rules select different collision amplitudes.

Phase shift and inherited interaction. The zero interaction above comes from its particular reset locations. Shift the second clock and take

$$F(x) = 2\psi(x) + 2\psi(2x + 1/4).$$

The slope is still $V = 6$, but the nonreset readings now contribute. The complete boundary data are

| $b$ | $C_b$ | $U_b$ | $F(b-)$ | $F(b+)$ |
|---|---|---|---|---|
| 0 | 1 | $-1/2$ | $1/2$ | $-3/2$ |
| $3/8$ | 1 | $-1/4$ | $3/4$ | $-5/4$ |
| $7/8$ | 1 | $3/4$ | $7/4$ | $-1/4$ |

At every reset the actual reading is $F(b) = U_b$. On the unshifted two-point grid, only zero is a sampled reset. Also $\psi(3/4) = 1/4$ and $\widetilde{B}_2(3/4) = -1/48$, so the three terms of (7) are

$$\underbrace{\frac{1}{2}}_{\text{sampled reset}} + \underbrace{2\left(-\frac{1}{4}\frac{1}{4} + \frac{3}{4}\frac{1}{4}\right)}_{\text{inherited interaction}} - \underbrace{3\left(\frac{1}{6} - \frac{1}{48} - \frac{1}{48}\right)}_{\text{slope contribution}} = \frac{1}{2} + \frac{1}{4} - \frac{3}{8} = \frac{3}{8}.$$

The actual samples are $F(0) = F(1/2) = -1/2$, giving sampled energy $1/4$. For an independent integral check, the three open segments are

$$F(x) = \begin{cases} 6x - 3/2, & 0 < x < 3/8, \\ 6x - 7/2, & 3/8 < x < 7/8, \\ 6x - 11/2, & 7/8 < x < 1. \end{cases}$$

Integrating their squares gives $27/128 + 13/32 + 1/128 = 5/8$. Thus the defect is $5/8 - 1/4 = 3/8$, as the boundary formula gives. Dropping the inherited term would give $1/8$. Its contribution comes from the two unsampled boundaries, through the nonreset readings there.

The unsampled resets retain nonzero background readings (Figure 11). Their boundary phases carry that interaction into the defect.

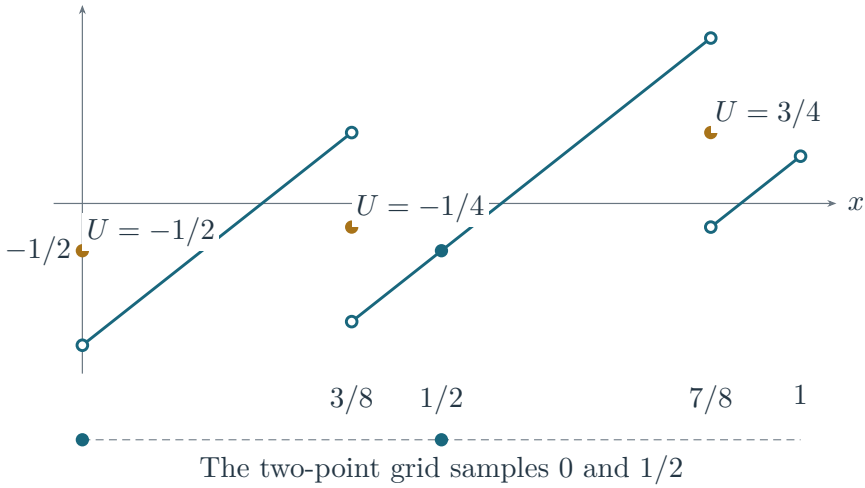



FIGURE 11. The shifted two-clock field has three resets, each of amplitude $C = 1$. Gold dots are their inherited readings $U$. Only the reset at zero is sampled, but the two unsampled resets still enter the boundary sum. The grid readings are both $-1/2$.

6.2. **Clock boundary calculation.** For a clock family with specified coefficients, list distinct reset locations before taking any squares. At each location, sum the coefficients of the resetting clocks to obtain $C_b$, and evaluate only the nonreset clocks to obtain $U_b$. The global slope $V$ is computed once. A boundary is sampled exactly when $qb - \theta$ is an integer. These data supply all three terms of (7).

This procedure also provides a useful diagnostic. If the amplitudes of two resetting clocks cancel, their joint point correction disappears, but their interactions at other boundaries must still be evaluated. Cancellation at one location does not remove either clock from the field.

6.3. **Reset-reading problem.**

**Problem 9** (A different reset reading)**.** Let $f = 2\psi$ and let $g = f$ away from zero, with $g(0) = 1$. Compute $D_q[g]$ and explain its sign.

**Problem 10** (Opposing resets)**.** Take two clocks with opposite coefficients, $f(x) = 2\psi(x)$ and $g(x) = -2\psi(2x)$, using the specified midpoint readings. Put $F = f + g$. List the component reset locations and their combined collision amplitudes. Compute $D_4[F]$ directly, then compare it with $D_4[f] + D_4[g]$. Determine the mixed contribution and explain the effect of the opposing coefficients at zero.

## 7. Interval pulses and endpoint conventions

A pulse asks a counting question. How many of the sample addresses fall inside its interval, and are the endpoints admitted? Moving an endpoint changes the integral continuously, but a sampled reading changes when the endpoint passes it. The boundary formula describes both effects using the same two endpoint locations.

The same theorem applies to a field with two boundaries and no rising clock segments. Fix $0 < \alpha < \beta < 1$ and $A \in \mathbb{C}$. Let $f = A$ on $(\alpha, \beta)$ and $f = 0$ outside $[\alpha, \beta]$. Specify $f(\alpha) = u$ and $f(\beta) = v$ independently.

**Proposition 7.1** (Pulse boundary identity)**.** *For every $q, \theta$,*

$$D_{q,\theta}[f] = \frac{|A|^2/2 - |u|^2}{q}\mathbf{1}_{\{q\alpha-\theta\in\mathbb{Z}\}} + \frac{|A|^2/2 - |v|^2}{q}\mathbf{1}_{\{q\beta-\theta\in\mathbb{Z}\}} + \frac{|A|^2}{q}\big(\psi(q\beta-\theta) - \psi(q\alpha-\theta)\big). \tag{8}$$

*For the half-open convention $u = A, v = 0$, this also equals*

$$|A|^2\left(\beta - \alpha - \frac{\lceil q\beta-\theta\rceil - \lceil q\alpha-\theta\rceil}{q}\right). \tag{9}$$

*Proof.* The energy jumps by $|A|^2$ at $\alpha$ and by $-|A|^2$ at $\beta$. All regular derivatives and derivative jumps vanish. Theorem 5.1 gives (8). Under the half-open convention the grid points in the pulse correspond exactly to integers $a$ with $q\alpha - \theta \le a < q\beta - \theta$. Their number is the ceiling difference in (9). □

For a fixed amplitude and fixed endpoint readings, the defect is $O(q^{-1})$ uniformly in the grid offset. A persistent defect can be produced by growing the amplitude. Let $q$ tend to infinity through multiples of three, take $\alpha = 1/3$, $\beta = 2/3$, $\theta = 0$ and $A_q = \sqrt{\lambda q}$ with

$\lambda \geq 0$. Both endpoints lie on the grid. The four conventions below have identical continuous energy.

| Endpoint readings | Interpretation | $D_q[f_q]$ |
|---|---|---|
| $(A_q, 0)$ | half-open pulse | $0$ |
| $(0, 0)$ | both boundaries omitted | $\lambda$ |
| $(A_q/2, A_q/2)$ | midpoint field readings | $\lambda/2$ |
| $(A_q, A_q)$ | both boundaries included | $-\lambda$ |

These values follow directly from (8). They show why a constant defect belongs to a specified family and reading convention. No universal nonzero constant is forced by discreteness alone.

Overlapping pulses also retain their interactions. For half-open intervals $I_j$, let $f = \sum_j A_j \mathbf{1}_{I_j}$. Then the energy contains $2\,\mathrm{Re}(A_j \overline{A_k})\mathbf{1}_{I_j \cap I_k}$ for each pair $j < k$. Formula (11) keeps these overlap contributions. Their boundaries may coincide even when the intervals themselves differ.

Figure 12 compares the four endpoint conventions. The open pulse has the same shape and integral in every row.

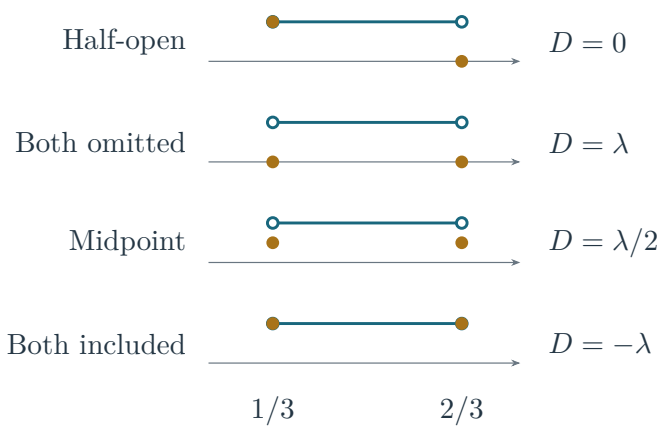



FIGURE 12. Endpoint choices for the same pulse. Gold dots show actual readings; the open segment has amplitude $A = \sqrt{\lambda q}$. When $3 \mid q$, both endpoints are sampled and these four defects are exact. The common continuous energy is unchanged. The profiles show the nonzero-amplitude case, divided by $A$.

7.1. **Overlap problem.**

**Problem 11** (The cost of dropping overlap)**.** Let $I = [1/4, 3/4)$, $J = [1/2, 1)$ on $\mathbb{T}$, and $f = \mathbf{1}_I + \mathbf{1}_J$. Compute $D_5[f]$ and compare it with the sum of the two individual defects.

**Problem 12** (Two adjacent pulses)**.** Put $f = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1)}$ on the periodic interval. For an even integer $q$, calculate $D_q[f]$. Change only the combined field's readings at 0 and $1/2$ to zero, obtaining $g$. Calculate $D_q[g]$ and identify the complete contribution of the replacement.

**Problem 13** (Two refinement sequences)**.** Take $f = \mathbf{1}_{[0,1/3)}$ and $f_q = \sqrt{q}\, f$ on the unshifted grid. Calculate both defects when $q = 3m$ and when $q = 3m + 1$. Does $D_q[f]$ tend to zero along both sequences? Does $D_q[f_q]$ have a common limit? Decide from the count, without a fixed-field error constant.

The ramp, resetting clocks and interval pulses now share a calculation. Each supplies its own readings and boundary data to the same formula. We can next change the field or its observation while keeping this account intact. The mixed terms that already appeared in overlapping pulses will accompany those changes.

Part 2

# Field and observation changes

The boundary formula compares two observations of a fixed field. We now let the field and the observation change. Adding a component creates mixed work; moving a window removes some readings and admits others. Unequal weights also change how cancellation is measured.

The two-clock tail, the moving signal and the overlapping pulses give complete examples of these changes. Together they establish the habits needed for a growing family. Expand the whole energy, retain the actual observation, and track how every error allowance depends on scale.

## 8. Boundary calculations

The choice of operation depends on the data available. Write the observed energies, then choose the formula whose required data can be evaluated or bounded.

8.1. **Readings and changed data.** Give the addresses, their masses and the actual readings. An integer tail beginning after $S$ starts at $S + 1$. Write a changed field as $f + g$ on the union of old and new addresses. The following choices determine which additional data the calculation needs.

| Task | First operation | Required data |
|---|---|---|
| Compare integral and grid | Boundary formula, Theorem 5.1 | Traces, jumps, regular derivatives and offset. |
| Bound weighted interaction | Summation by parts, (10) | Primitive bound and variation of the weighted partner. |
| Change field and observation | Energy-change identity, Proposition 3.1 | Old and new masses, inherited field and increment. |
| Restrict additions | Invariant readings, Theorem 9.3 | Allowed additions and a preserved linear reading. |

An energy change may first be decomposed and its mixed term then bounded by summation by parts. The weighted calculation follows.

For the phase-shifted two-clock field in Section 6, the three boundary readings determine the defect without estimating a long sum. The weighted tail developed in Section 10 has infinitely many addresses. Its useful input is a running sum, or *primitive*, and the variation of the sequence paired with it. The observations determine which calculation can be completed.

8.2. **Observation weights and primitive sums.** Cancellation can be visible in a running total even when the individual terms are large. For $u(1), \dots, u(4) = (1, -1, 1, -1)$, the running totals are $1, 0, 1, 0$. Against arbitrary readings $x_1, \dots, x_4$,

$$\sum_{n=1}^{4} u(n)x_n = (x_1 - x_2) + (x_3 - x_4).$$

Constant partners cancel completely. Slowly changing partners leave only their changes. Summation by parts extends this calculation to any sequence, with its final running total supplying an endpoint term.

On a uniform periodic grid, the Bernoulli kernels in Theorem 5.1 move the calculation onto point readings, jumps and regular derivatives. For an arbitrary weighted sequence, ordinary summation by parts is the corresponding operation. If $T(n) = \sum_{k=a}^{n} u(k)$, $T(a-1) = 0$, and $x_n$ is any complex sequence, then finite telescoping gives

$$\sum_{n=a}^{b} u(n)x_n = T(b)x_b + \sum_{n=a}^{b-1} T(n)(x_n - x_{n+1}). \tag{10}$$

Indeed, substitute $u(n) = T(n) - T(n-1)$ and shift the second finite sum; its lower endpoint vanishes because $T(a-1) = 0$. For mixed energy take $x_n = v_n\overline{g(n)}$, where $v_n$ is the observation mass. Its difference is

$$x_n - x_{n+1} = (v_n - v_{n+1})\overline{g(n)} - v_{n+1}\overline{g(n+1) - g(n)}.$$

Both the mass variation and the field variation remain.

With an unanchored primitive $F$, the lower endpoint is $-F(a-1)x_a$. Subtracting $F(a-1)$ from the primitive redistributes that

endpoint across the formula. It cannot remove a contribution to the observed energy. At an infinite upper endpoint, write the finite identity first and prove that its endpoint tends to zero.

In Figure 13, the alternating readings have a small running sum. Summation by parts charges that sum to changes in the partner, including the endpoint.

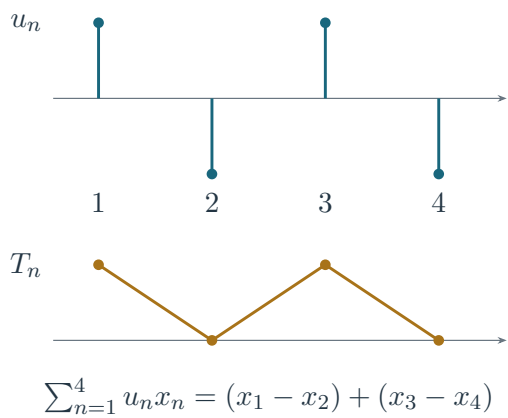



FIGURE 13. The running sum retains the cancellation before an estimate is made. For $(1, -1, 1, -1)$ it stays between zero and one. Pairing the sequence with another table reads differences of that table. The partner's variation determines how much of the cancellation survives.

8.3. **Uniform estimates and signed work.** If $|T(n)| \le P$, (10) gives

$$\left| \sum_{n=a}^{b} u(n) x_n \right| \le P \left( |x_b| + \sum_{n=a}^{b-1} |x_n - x_{n+1}| \right).$$

The primitive bound measures cancellation in $u$; the bracket measures the variation of its weighted partner. For a growing family, write both as functions of its scale. A small primitive for each component is useful only after the number, amplitudes and interactions of those components have also been counted.

If this absolute estimate is too large, return to the signed identity. The endpoint and interior terms may cancel. In an energy change, a removed part of the observation contributes negatively, and mixed

work can offset a new square. Combine these contributions before deciding which absolute bounds the calculation needs.

**Problem 14** (Moving an endpoint)**.** Observe a fixed real field on $a < n \le b$ with masses $v_n > 0$. Move the window to $a+1 < n \le b+1$ and add a field $g$. Write the change without losing either endpoint.

**Problem 15** (Weight variation with a constant partner)**.** On $1 \le n \le 5$, take $u(n) = (-1)^{n+1}$, $v(n) = 1$ and $w_n = 1/[n(n+1)]$. Compute the weighted sum directly. Recompute it by summation by parts, using the primitive $T(n) = \sum_{k=1}^{n} u(k)$. Retain the upper endpoint and separate variation of $w_n$ from variation of $v(n)$. Identify the errors in arguments that discard either the endpoint or all interior terms because the partner $v$ is constant.

## 9. Interaction, refinement and dilation

The opening calculation showed that adding two fields also adds their interaction. We now make that interaction an operation, then track what happens when the sampling grid changes.

For admissible fields $f, g$ on a common boundary partition, define

$$K_{q,\theta}(f,g) = \int_0^1 f\overline{g}\,dx - Q_{q,\theta}(f\overline{g}).$$

For real fields this is simply the discrepancy of their pointwise product. For complex fields the conjugate gives $K(g,f) = \overline{K(f,g)}$, with the grid subscripts understood. The form is linear in its first input; scaling its second input by $c$ scales the result by $\overline{c}$. These properties are called Hermitian symmetry and sesquilinearity. The mixed form need not be positive. Linearity gives

$$D_{q,\theta}[f+g] = D_{q,\theta}[f] + D_{q,\theta}[g] + 2\,\mathrm{Re}\,K_{q,\theta}(f,g). \tag{11}$$

Theorem 5.1 applies directly to $f\overline{g}$, so it also resolves each mixed contribution into boundary and regular terms.

9.1. **Collision matrix.** With many components, repeating a two-field expansion becomes cumbersome. A matrix stores the result of every pairwise comparison. Its diagonal records individual defects, and its off-diagonal entries record the mixed defects. Choosing the component amplitudes then selects one quadratic expression from

this table. For two identical real fields of defect $d$, all four entries are $d$; amplitudes $(1,1)$ give $4d$, while $(1,-1)$ give zero.

Let $f_1, \dots, f_s$ be admissible. For $z \in \mathbb{C}^s$, put $F_z = \sum_{j=1}^{s} z_j f_j$ and define

$$G^{\mathrm{cont}}_{jk} = \int_0^1 \overline{f_j} f_k \, dx, \qquad G^{\mathrm{grid}}_{jk} = Q_{q,\theta}(\overline{f_j} f_k), \qquad H = G^{\mathrm{cont}} - G^{\mathrm{grid}}.$$

Both Gram matrices are positive semidefinite. Their difference can have either sign.

**Proposition 9.1** (Matrix form and change of coordinates)**.** *For every coefficient vector $z$,*

$$D_{q,\theta}[F_z] = z^* H z.$$

*If every $f_j$ takes its midpoint reading at each boundary, the matrix of the sampled point correction is*

$$P_{jk} = \frac{1}{4q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} \overline{[f_j]_b} [f_k]_b.$$

*In particular, $P$ is positive semidefinite and*

$$z^* P z = \frac{1}{4q} \sum_{b \in \mathcal{B} \cap X_{q,\theta}} \left| \sum_j z_j [f_j]_b \right|^2 .$$

*A unitary change of coefficients $z = Uw$ replaces $H$ and $P$ by $U^*HU$ and $U^*PU$. It preserves their eigenvalues and traces.*

*Proof.* Expand $|F_z|^2$ before integrating or sampling. This gives $z^*Hz$. Apply the midpoint identity to $F_z$ at each sampled boundary to obtain $z^*Pz$. Its displayed expression is a sum of nonnegative squares. Substituting $z = Uw$ gives the transformed matrices. Since $U^*U = I$, they are unitarily similar to the original matrices, which proves the last assertion. □

Midpoint readings make the collision point correction nonnegative. The rest of the defect is the signed matrix $H - P$. A basis change reorganizes both parts, but a bound for the remainder still requires an estimate. A coordinate change preserves the energy of a fixed field; different coefficient directions can still carry different energies. Character transforms can be used as unitary coordinate changes when their normalization and underlying finite space are specified; additional symmetry is needed for any stronger conclusion.

9.2. **Grid refinement and dilation.** The six-point grid separates into two three-point grids. Its even indices give $0, 1/3, 2/3$, and its odd indices give $1/6, 1/2, 5/6$. The six-point average is therefore the mean of the two three-point averages. This is the first identity below.

Dilation changes the arguments read by the grid. Sampling $f(2x)$ at those same six addresses reads $f$ at $0, 1/3, 2/3$, each twice. Only three distinct readings remain. In general the greatest common divisor of the dilation and grid size counts these repetitions.

Figure 14 shows the partition of the six-point observation. Dilation must additionally keep track of repeated arguments and the resulting offset.

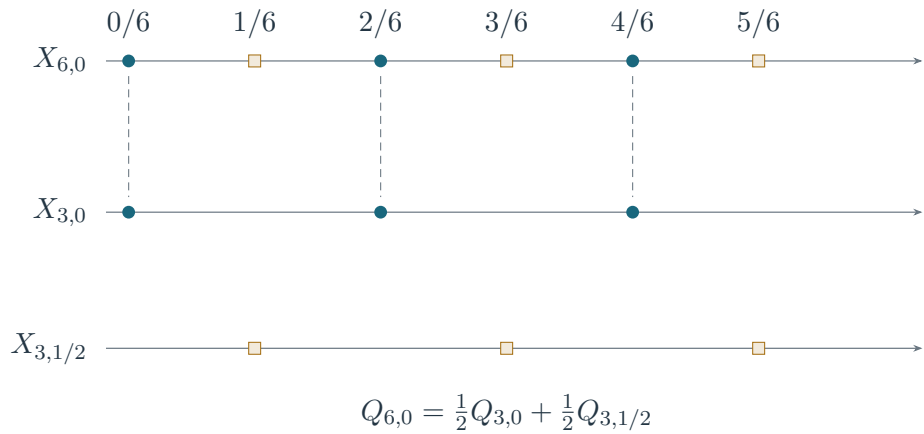



FIGURE 14. Refinement splits the six-point grid into two offset three-point grids. Blue circles mark even indices and gold squares mark odd indices. Each original mass $1/6$ is half a three-point mass. The field is held fixed throughout this identity.

**Proposition 9.2** (Grid refinement and dilation)**.** *For a fixed field $f$, positive integers $r, m, q$ and $0 \le \theta < 1$,*

$$D_{rq,\theta}[f] = \frac{1}{r}\sum_{j=0}^{r-1} D_{q,(j+\theta)/r}[f]. \tag{12}$$

*For $\phi \in \mathbb{T}$, define $(U_{m,\phi}f)(x) = f(mx + \phi)$. If $d = (m, q)$, $q' = q/d$ and $m' = m/d$, then*

$$D_{q,\theta}[U_{m,\phi}f] = D_{q',\{m'\theta+q'\phi\}}[f]. \tag{13}$$

*These identities also hold for any measurable $f$ with finite squared integral and specified finite point readings.*

*Proof.* The $rq$ samples split into the $r$ classes with indices $ra + j$. Their averages are the $q$-point averages in (12). For dilation, $a \mapsto m'a$ permutes the residues modulo $q'$, each occurring $d$ times as $a$ ranges modulo $q$. The shifted arguments are therefore the $q'$-point grid with offset $\{m'\theta + q'\phi\}$. The continuous integral of $|f(mx+\phi)|^2$ equals that of $|f|^2$ by periodicity and substitution. □

For a fixed field, averaging over all grid offsets also gives

$$\int_0^1 D_{q,\theta}[f]\, d\theta = 0.$$

Indeed the substitution $x = (a + \theta)/q$ turns the offset average of $Q_{q,\theta}|f|^2$ into its continuous integral. A persistent defect along a specified aligned grid is compatible with this identity.

As a direct check on the role of resonance, take $F(x) = 2c\psi(mx + \phi)$. Put $\alpha = \{m'\theta + q'\phi\}$ as above. For the unit-frequency field $2c\psi$, the only boundary has $U_0 = 0$, $C_0 = c$ and $V = 2c$. Thus

$$D_{q,\theta}[F] = |c|^2 \left( \frac{d}{q} \mathbf{1}_{\{\alpha=0\}} - \frac{4d^2}{q^2} \widetilde{B}_2(\alpha) \right). \tag{14}$$

For $q = m$ and zero phases this is $|c|^2/3$, since every sample is a reset. For fixed $m, c, \phi$ it tends to zero as $q \to \infty$.

For a unit clock with $c = 1$, the unshifted and half-cell formulas are particularly simple:

$$D_q[2\psi] = \frac{1}{q} - \frac{2}{3q^2}, \qquad D_{q,1/2}[2\psi] = \frac{1}{3q^2}.$$

At offset $\theta = 1/8$, the same field instead has defect $-11/(48q^2)$. These expressions all tend to zero, while having different signs and different leading orders. The jump at an exactly aligned offset is the contribution of its specified reset reading.

**Problem 16** (Dilation of a shifted grid)**.** Let $f = 2\psi$ and $f_r(x) = f(rx)$. On the six-point grid with $\theta = 1/2$, list the distinct arguments of $f$ and their multiplicities for $r = 2$ and $r = 4$. Compute both energy defects. Repeat for $\theta = 0$. Explain why the common value $\gcd(r, 6) = 2$ does not by itself determine the defect on the shifted grid.

9.3. **Joint field and grid refinement.** Equation (12) holds with the field fixed. Suppose the new field is $f + g$ and the new grid has $rq$ points. Combining refinement with polarization gives

$$D_{rq,\theta}[f+g] = \frac{1}{r}\sum_{j=0}^{r-1} D_{q,(j+\theta)/r}[f] + D_{rq,\theta}[g] + 2\operatorname{Re} K_{rq,\theta}(f,g).$$

The added field contributes its own defect and its interaction with the old field. This is the appropriate identity when refinement adds clocks or changes amplitudes. A grid identity alone would omit those terms.

9.4. **Algebraic rules.** Fix the grid and write $D$ and $K$ with its subscripts suppressed. For scalars $c, d$ and admissible fields,

$$\begin{aligned} D(cf) &= |c|^2 D(f), \\ K(cf, dg) &= c\overline{d}\, K(f,g), \\ D(f+g) &= D(f) + D(g) + 2\operatorname{Re} K(f,g). \end{aligned}$$

These follow by expanding the squared magnitude or mixed product before either observation. Point changes use Proposition 4.1. Grid refinement and dilation use Proposition 9.2; translation is its case $m = 1$, with new offset $\{\theta + q\phi\}$. Write

$$P_{q,\theta}(f) = A_{q,\theta}(|f|^2), \qquad R_{q,\theta}(f) = D_{q,\theta}[f] - P_{q,\theta}(f).$$

Then $D = P + R$ separates the sampled point contribution from the remaining signed work. Each rule keeps the same field, grid and reading conventions unless their change is explicitly included.

9.5. **Coordinate and resonance problems.**

**Problem 17** (Two identical fields in a different basis)**.** Take $f_1 = f_2 = 2\psi$, sampled on the unshifted $q$-point grid. Find the defect matrix and diagonalize it with

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

**Problem 18** (Resonance and one extra sample)**.** For $f_m(x) = 2\psi(mx)$, compare the unshifted grids with $q = m$ and $q = m + 1$ as $m \to \infty$.

9.6. **Invariant readings.** The preceding identities follow energy when a field or grid changes. There is another useful question at a fixed observation. If only certain field additions are allowed, which readings can they never change? Such a reading forces some energy to remain, however the allowed coefficients are chosen. A lower bound of this kind will later control the negative shell term in a complete energy increment.

An addition can change every sample while leaving one combination of samples fixed. That fixed reading can force energy to remain. Take three equally weighted addresses and the field $f = (0, 1, 0)$. Permit any affine addition $h(i) = a + bi$. The second difference

$$L(f) = f(1) - 2f(2) + f(3)$$

annihilates every such addition, so $L(f + h) = -2$. Cauchy–Schwarz gives

$$|L(f + h)|^2 \le (1^2 + (-2)^2 + 1^2) \sum_{i=1}^{3} |f(i) + h(i)|^2.$$

Consequently the observed energy is at least $2/3$. The constant addition $h = -1/3$ attains that value, leaving $(-1/3, 2/3, -1/3)$. The permitted additions can remove the affine part of the readings; the remaining bend has a definite energy cost.

The two tables in Figure 15 retain the same bend. Removing their common level reduces energy as far as the affine additions permit.

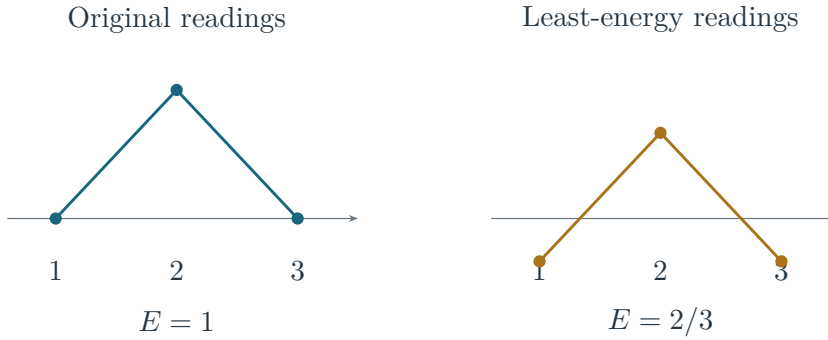



FIGURE 15. An affine addition changes the level and tilt but leaves the second difference fixed. Subtracting $1/3$ attains the least possible energy in this example. The retained bend forces energy $2/3$ under unit weights. Segments join the readings to show their shape.

Here the weights are one, so energy is a sum rather than a normalized mean. Changing the weights changes the cost of preserving the same reading. With weights $(1, 2, 1)$, the lower bound becomes one and is attained by the addition $h = -1/2$.

**Theorem 9.3** (Energy forced by invariant readings)**.** *Let $I$ be a finite address set, let $w_i > 0$, and write $E_w[f] = \sum_{i\in I} w_i |f(i)|^2$. Let $V$ be a linear space of allowed additions. Suppose*

$$L(f) = \sum_{i\in I} a_i f(i), \qquad L(h) = 0 \quad (h \in V), \qquad c_L = \sum_{i\in I} \frac{|a_i|^2}{w_i} > 0.$$

*Then, for every $h \in V$,*

$$E_w[f + h] \geq \frac{|L(f)|^2}{c_L}. \tag{15}$$

*More generally, if nonzero functionals $L_1, \dots, L_k$ annihilate $V$ and have pairwise disjoint supports $I_1, \dots, I_k$, then*

$$E_w[f + h] \geq \sum_{\alpha=1}^{k} \frac{|L_\alpha(f)|^2}{\sum_{i\in I_\alpha} |a_{\alpha i}|^2 / w_i}. \tag{16}$$

*Proof.* The invariant reading is $L(f+h) = L(f)$. Apply Cauchy–Schwarz to $a_i/\sqrt{w_i}$ and $\sqrt{w_i}(f(i)+h(i))$ to obtain (15). For the second assertion apply the same argument on each support. Disjointness allows their energies to be added, and the energy on any unused address is nonnegative. □

This is weighted Cauchy–Schwarz used as a refinement rule. First specify the permitted additions, then construct a reading that all of them annihilate. Its value is inherited from the old field. The denominator records how much energy is required to carry that value under the actual observation weights. No sign condition is imposed on the individual field components.

9.7. **Joint invariant readings.** Several invariant readings may use the same addresses. Their interaction must then be retained, just as it is for the original fields. The task is to find the least energy compatible with all the retained readings at once. Let the rows of a matrix $A$ be their coefficients, so $Af$ is the list of retained values, and let $W = \operatorname{diag}(w_i)$ record the cost of each address. The condition $\ker A = V$ says that these readings detect every change outside the allowed space. When some such changes remain undetected, the same formula still gives a lower bound, but need not be attained by an allowed addition.

**Proposition 9.4** (Joint invariant readings)**.** *Suppose the rows of $A$ are linearly independent and $Ah = 0$ for every $h \in V$. Put $G = AW^{-1}A^*$ and $b = Af$. Then $G$ is positive definite and*

$$E_w[f+h] \geq b^*G^{-1}b \qquad (h \in V). \tag{17}$$

*If $\ker A = V$, the right side is the minimum energy over all allowed additions. For disjoint supports $G$ is diagonal, and this bound is (16).*

*Proof.* For nonzero $c$, row independence gives $A^*c \neq 0$. Hence

$$c^*Gc = (A^*c)^*W^{-1}(A^*c) > 0.$$

Define $z = W^{-1}A^*G^{-1}b$. Then $Az = b$. For any $h \in V$, put $r = f + h - z$. Since $Ar = 0$, one has $z^*Wr = (G^{-1}b)^*Ar = 0$. Thus

$$E_w[f+h] = E_w[z] + E_w[r] = b^*G^{-1}b + E_w[r].$$

This proves the bound. If $\ker A = V$, then $z - f \in V$ and equality is attained by that addition. Disjoint supports make all off-diagonal entries of $G$ vanish. □

This is the classical minimum-norm problem under linear constraints [5, Sections 6.2 and 10.1.1], written in the coordinates of the invariant readings. The joint matrix measures their overlap in the reciprocal weights. Adding their separate lower bounds without this matrix can count the same retained energy twice.

**Problem 19** (Two overlapping second differences)**.** Give four addresses unit weights, take $f = (0, 1, 0, 0)$, and permit all affine additions. Use

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

to find the minimum energy. Compare it with the sum of the two separate reading bounds.

**Problem 20** (Weighted adjustment)**.** Give three addresses weights $(1, 2, 1)$ and field $f = (1, 0, 0)$. The permitted additions are constant fields. Find the minimum energy and an addition attaining it. Then obtain the same value from two independent invariant readings and their joint matrix.

**Problem 21** (A reading that misses the residual)**.** Give three addresses unit weights, take $f = (0, 1, 0)$, and again permit constant additions. The reading $L(v) = v_1 - v_3$ is invariant and $L(f) = 0$. Does its lower bound determine the minimum energy? Calculate that minimum and explain what the reading fails to observe.

## 10. Native weights and moving fields

A complete observation can assign positive mass to infinitely many addresses and still have finite total mass. Its unresolved tail is then a measured part of the same whole. The reciprocal weights introduced in Section 2.5 let us evaluate that mass and follow the energy when both the field and the first admitted address change.

An equally weighted periodic grid and an activated arithmetic tail observe a field in different ways. Summation by parts remains the classical tool. The task here is to carry the actual observation, reset

reading and field change through it. Here *native* means that the weights are prescribed by the arithmetic observation.

The first three native masses are $1/2, 1/6, 1/12$. The later readings continue indefinitely, with decreasing mass. A constant unit field has energy one over all positive addresses, but energy $1/2$ after the first address is removed. That change occurs before the field itself has changed. The weights are

$$w_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \sum_{n \geq a} w_n = \frac{1}{a}.$$

For an integer $S \geq 0$, define the native energy and mixed observation

$$E_S[p] = \sum_{n>S} w_n |p(n)|^2, \qquad \langle p, v \rangle_S = \sum_{n>S} w_n p(n) \overline{v(n)}.$$

The cutoff is part of the observation. It excludes $S$ and begins at $S+1$. A finite clock field is bounded, so its energy is finite for every fixed cutoff. No uniform bound in a changing clock family follows from this assertion alone.

The weights also have an exact geometric reading. For the step interpolation $\widehat{p}(u) = p(\lfloor u \rfloor)$,

$$E_S[p] = \int_{S+1}^{\infty} \frac{|\widehat{p}(u)|^2}{u^2}\, du = \int_0^{1/(S+1)} |p(\lfloor 1/x \rfloor)|^2\, dx. \tag{18}$$

Each address cell has mass $\int_n^{n+1} u^{-2}\, du = w_n$. Thus the infinite tail consists of reciprocal cells accumulating at zero. Formula (18) is an identity of observations, not a limiting approximation by equally weighted samples.

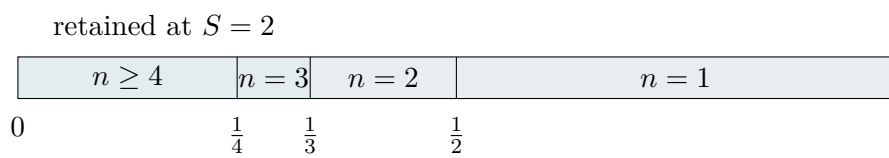



FIGURE 16. Reciprocal cells turn an infinite weighted observation into an integral over a finite interval. The blue portion retains all addresses $n \geq 3$. Its length is exactly their native mass $1/3$.

The cell $(1/(n+1), 1/n]$ has length $w_n$. At $S = 2$, the retained cells $n \geq 3$ fill the interval $0 < x \leq 1/3$. The entire infinite tail is present in this finite interval; its cells become progressively shorter near zero. Individual endpoints have no effect on the integral in (18).

10.1. **Approximation of unity.** The reciprocal cells make the unity account of Section 2.5 geometric. They partition $(0, 1]$, and $\mathbf{1}(n) = 1$ is the constant reading on every cell. Thus $E_0[\mathbf{1}] = 1$, whereas $E_S[\mathbf{1}] = 1/(S+1)$. Activation retains the mass of the selected tail. Renormalizing that tail would change the observation.

For $N \geq 2$, let a clock approximation and its residual be

$$p_N(n) = \sum_{j=2}^{N} c_{N,j}\{n/j\},$$

$$r_N(n) = \mathbf{1}(n) - p_N(n), \qquad \mathcal{R}_N = E_0[r_N].$$

The coefficients may change as more clocks become available, while unity remains the same target. Every such finite field is bounded, so its complete error is finite. At a common multiple of its periods, all fractional clocks read zero. The residual there reads one and has positive native mass. Thus every finite clock approximation leaves positive error.

For an added finite clock field $h$, the new residual is $r_N - h$. Expanding its square gives the complete error change

$$E_0[r_N - h] - E_0[r_N] = E_0[h] - 2\,\mathrm{Re}\langle r_N, h\rangle_0.$$

All terms converge by boundedness and summability of the weights. An improvement requires the residual's signed response to pay for the correction's energy. Several corrections must be combined before their cost is squared. If the old and new coefficients minimize error over nested clock spaces, the minimum cannot increase, because the old fit is still available. A prescribed coefficient change must instead be assessed by the displayed identity.

The first fractional clock. The clock $\phi_2(n) = \{n/2\}$ reads $1/2$ on odd addresses and zero on even addresses, including every reset. The odd addresses have mass $\log 2$. To see the infinite sum directly, for $M \geq 1$ write

$$\sum_{k=1}^{M} w_{2k-1} = \int_0^1 \frac{1 - x^{2M}}{1+x}\,dx \longrightarrow \int_0^1 \frac{dx}{1+x} = \log 2.$$

The omitted integral is nonnegative and at most $1/(2M+1)$. The even addresses therefore have mass $1-\log 2$. For a real coefficient $c$, the complete squared error is

$$E_0[\mathbf{1} - c\phi_2] = \log 2\,(1 - c/2)^2 + 1 - \log 2.$$

It is minimized at $c = 2$. The fit equals unity at every odd address; its residual equals unity at every even address. Their energies are $\log 2$ and $1 - \log 2$. This is a partition of the original target over the complete observation, with actual right-hand clock readings at every reset [22].

Figure 17 shows which readings the first clock resolves. Both parts extend through the entire native observation.

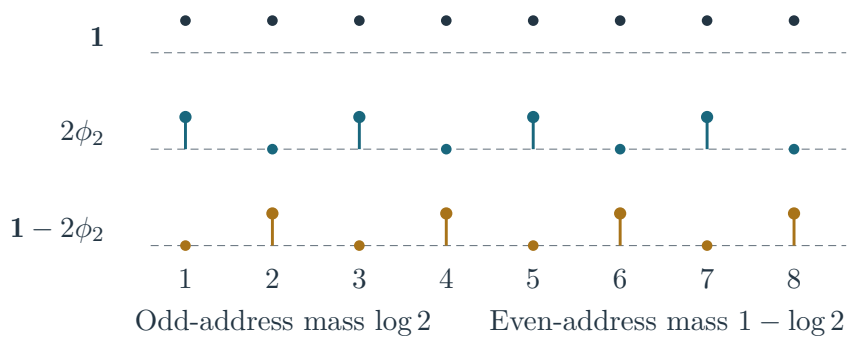



FIGURE 17. Twice the first fractional clock resolves unity on every odd address and leaves it unresolved on every even address. The displayed pattern repeats indefinitely. The two complete energies are the native masses $\log 2$ and $1 - \log 2$, so the omitted addresses remain part of the same observation.

These comparisons keep the observation $E_0$ fixed. A moving activation cutoff also removes old readings; Proposition 10.3 retains that additional contribution.

10.2. **Reciprocal boundary terms.** For the constant reading one, the identity $w_n = 1/n - 1/(n+1)$ makes successive terms cancel. Between $a$ and $N$ only $1/a - 1/(N+1)$ remains. A changing reading interrupts that cancellation by its successive differences. The next formula retains those interruptions and both endpoints.

Write $\Delta h(n) = h(n) - h(n-1)$ for a backward difference.

**Proposition 10.1** (Reciprocal boundary identity)**.** *For arbitrary complex readings $h(a), \dots, h(N)$, with $1 \le a \le N$,*

$$\sum_{n=a}^{N} w_n h(n) = \frac{h(a)}{a} - \frac{h(N)}{N+1} + \sum_{n=a+1}^{N} \frac{\Delta h(n)}{n}. \tag{19}$$

*If $\sum_{n \ge a} w_n |h(n)| < \infty$ and $h(N)/(N+1) \to 0$, then*

$$\sum_{n=a}^{\infty} w_n h(n) = \frac{h(a)}{a} + \lim_{N \to \infty} \sum_{n=a+1}^{N} \frac{\Delta h(n)}{n}. \tag{20}$$

*Bounded $h$ satisfies both hypotheses. The boundary series is taken in increasing address order and need not converge absolutely.*

*Proof.* Expand $w_n = 1/n - 1/(n+1)$ and shift the index in the second finite sum. The lower endpoint is $h(a)/a$, the upper endpoint is $-h(N)/(N+1)$, and the interior terms are the displayed differences. Under the stated hypotheses, the left side converges absolutely and the upper endpoint tends to zero. The finite equality then proves existence and the value of the ordered limit on the right. □

For energy, put $h = |p|^2$ and $a = S + 1$. A bounded clock field therefore has a complete boundary account beginning with $|p(S+1)|^2/(S+1)$. The combined differences $\Delta|p|^2$ must be kept together in the ordered limit. Separating a positive jump-square series from its mixed series can produce two divergent sums even when their joined series converges.

For interaction estimates it is more useful to move the difference onto the weighted partner. Define the anchored primitive $T_a(n) = \sum_{k=a}^{n} u(k)$, with $T_a(a-1) = 0$.

**Proposition 10.2** (Weighted variation identity)**.** *For $1 \le a \le N$,*

$$\sum_{n=a}^{N} w_n u(n) v(n) = T_a(N) w_N v(N) + \sum_{n=a}^{N-1} T_a(n)\big((w_n - w_{n+1}) v(n) - w_{n+1} \Delta v(n+1)\big), \tag{21}$$

*where*

$$w_n - w_{n+1} = \frac{2}{n(n+1)(n+2)}.$$

*Suppose* $|T_a(n)| \le B$, $|v(n)| \le V$ *and* $\sum_{n\ge a} w_{n+1}|\Delta v(n+1)| < \infty$. *Then the upper endpoint tends to zero, both infinite sums in* (21) *converge absolutely, and*

$$\left|\sum_{n\ge a} w_n u(n)v(n)\right| \le B\left(Vw_a + \sum_{n\ge a} w_{n+1}|\Delta v(n+1)|\right). \tag{22}$$

*Proof.* Set $x_n = w_n v(n)$ and substitute $u(n) = T_a(n) - T_a(n-1)$ into the finite sum. Index shifting gives $T_a(N)x_N + \sum_{n=a}^{N-1} T_a(n)(x_n - x_{n+1})$. Expansion of $x_n - x_{n+1}$ gives the two terms in (21). If instead $F(n) = \sum_{k=1}^{n} u(k)$ is used, the finite formula includes the lower endpoint $-F(a-1)x_a$:

$$\sum_{n=a}^{N} u(n)x_n = F(N)x_N - F(a-1)x_a + \sum_{n=a}^{N-1} F(n)(x_n - x_{n+1}).$$

Anchoring replaces $F$ by $F - F(a-1)$; it does not discard that endpoint.

For the limit, $|T_a(N)x_N| \le BVw_N \to 0$ and $\sum_{n\ge a}(w_n - w_{n+1}) = w_a$. These facts and the assumed weighted variation prove absolute convergence of the right side and the stated bound. Also $|u(n)| \le 2B$ (and $|u(a)| \le B$), so the original weighted product converges absolutely as well. □

For a conjugate mixed observation, replace $v$ by $\overline{v}$. The first interior term is caused by variation of the weights. It is present even when the partner field is constant.

10.3. **Right-hand clocks and combined resets.** At integer addresses, the clock $\{u/3\}$ repeats the readings $0, 1/3, 2/3$. At three it resets to zero, while its left trace is one. A sum of such clocks is specified by a coefficient table, called its source. Here $q(j)$ is the coefficient of the clock of length $j$; it is a function of $j$, distinct from the grid size $q$ used earlier. At address $n$, precisely the lengths dividing $n$ reset together.

Let $q(j)$, $j \ge 1$, be a finitely supported complex source. Define

$$p(u) = \sum_{j} q(j)\{u/j\}, \qquad \sigma = \sum_{j} \frac{q(j)}{j}, \qquad c(n) = \sum_{j|n} q(j).$$

At an integer address $n$, the actual reading is the right-hand value. The traces and discrete increment satisfy

$$p(n) = p(n+), \qquad p(n-) = p(n)+c(n), \qquad \Delta p(n) = \sigma - c(n). \tag{23}$$

Indeed each fractional clock rises by $1/j$ between successive integer addresses and subtracts one exactly when $j \mid n$. If the source is reciprocally neutral, $\sigma = 0$. The slope then vanishes on every open unit cell, $p(n-) = p(n-1)$, and the combined signed reset is $\gamma(n) = -c(n)$.

For a neutral source the complete energy jump is

$$\Delta|p(n)|^2 = 2\,\mathrm{Re}\big(\overline{p(n-1)}\gamma(n)\big) + |\gamma(n)|^2. \tag{24}$$

The divisor contributions are added in $c(n)$ before taking a square. Its square contains every interaction between simultaneous resets. The other term is their work against the inherited reading. Substituting their joined expression into (20) preserves the ordered boundary account.

The same rule for two complex fields is

$$\begin{aligned}\Delta(X\overline{Y})(n) = {} & X(n-1)\overline{\Delta Y(n)} + \overline{Y(n-1)}\Delta X(n) \\ & + \Delta X(n)\overline{\Delta Y(n)}.\end{aligned}$$

It retains both inherited readings and the simultaneous jump product. All these identities follow by expanding the new reading as the old reading plus its increment.

10.4. **Point-value correction at resets.** For two sequences with finite native energy, the point-value replacement rule is

$$E_S[g] - E_S[f] = \sum_{n>S} w_n\big(|g(n)|^2 - |f(n)|^2\big). \tag{25}$$

Only changed readings contribute. The series is absolutely convergent, since its absolute terms are bounded by the two energy series. This extends Proposition 4.1 to the weighted infinite set. The sign is the change in sampled energy; it is opposite to the change in an integral-minus-sample defect when the integral is fixed.

At a reset put $m_n = (p(n+) + p(n-))/2$ and $\gamma_n = p(n+) - p(n-)$. The actual right reading is $m_n + \gamma_n/2$, so

$$|p(n+)|^2 - |m_n|^2 = \operatorname{Re}(m_n\overline{\gamma_n}) + \frac{1}{4}|\gamma_n|^2, \tag{26}$$

$$a_n^+(|p|^2) := \frac{|p(n-)|^2 + |p(n+)|^2}{2} - |p(n+)|^2 = -\operatorname{Re}(m_n\overline{\gamma_n}). \tag{27}$$

The midpoint correction $|\gamma_n|^2/4$ has been adjusted by the change in the squared reading. The resulting right-hand correction can have either sign. For a neutral finite clock field, both the midpoint and right-hand sequences are bounded, so (25) applies to all their reset addresses. The midpoint is formed from the same clock field, not by silently replacing its fractional clocks with centered clocks.

Activation has its own boundary. If $b_S(n) = \mathbf{1}_{\{n>S\}}p(n)$, then for $n \geq 1$

$$\Delta b_S(n) = \mathbf{1}_{\{n=S+1\}}p(S+1) + \mathbf{1}_{\{n\geq S+2\}}\Delta p(n). \tag{28}$$

The first admitted reading is not generally the raw clock's reset at that address. The activation term records the whole admitted field.

10.5. **Field increments and activation.** There are two changes to follow. The moved cutoff removes old readings on $S < n \leq T$. Beyond $T$, the old field is still present when the addition arrives. Subtracting the removed energy and expanding the retained square gives the complete change. In particular, a positive new square can be outweighed by the removed shell or a negative interaction.

**Proposition 10.3** (Complete moving-field work)**.** *Let $0 \leq S \leq T$ be integers, let $E_S[p]$ and $E_T[p+v]$ be finite, and interpret all fields on the common tail $n > T$. Then*

$$\begin{aligned} E_T[p+v] - E_S[p] = &-\sum_{S<n\leq T} w_n|p(n)|^2 \\ &+\sum_{n>T} w_n\left(2\operatorname{Re}(\overline{p(n)}v(n)) + |v(n)|^2\right). \end{aligned} \tag{29}$$

*All terms converge absolutely. If $v = d + e$ with $E_T[d], E_T[e] < \infty$, its tail integrand is*

$$2\operatorname{Re}(\overline{p}d) + 2\operatorname{Re}(\overline{p}e) + |d|^2 + 2\operatorname{Re}(\overline{d}e) + |e|^2. \tag{30}$$

*Proof.* Split the old energy into $S < n \le T$ and $n > T$. On the common tail, expand $|p+v|^2 - |p|^2$. Since $|v|^2 \le 2|p+v|^2 + 2|p|^2$, the change $v$ has finite tail energy. Cauchy–Schwarz makes the mixed product absolutely summable. The second expansion follows by substituting $v = d + e$. The same Cauchy–Schwarz argument proves convergence of each separate term. □

The removed interval $S < n \le T$ is the *parent shell*. Its negative contribution contains the square of the entire old field. If that field is a sum of families, their shell interactions are inside the same square. If the change is a sum, its square likewise contains all interactions between changed families. A shorter observation window or a separate bound for the new square does not replace (29).

10.6. **Two-clock native tail.** Consider

$$p(u) = \{u\} - 2\{u/2\}.$$

Its reciprocal source sum is $1-2/2 = 0$. At integer addresses, $p(n) = -1$ for odd $n$ and $p(n) = 0$ for even $n$. The formal clock of length one has zero reading at every integer, but its source coefficient is needed for neutrality and for the reset formula.

At address one the left and right readings are $0, -1$. The midpoint reading is $-1/2$, with midpoint energy correction $1/4$. The actual right-hand correction is $1/2-1 = -1/2$. At address two the readings are $-1, 0$ and the right-hand correction is $1/2$. Thus positive midpoint corrections cannot be used unchanged even in this two-clock example.

The complete energy is

$$E_0[p] = \sum_{r\ge 0} \frac{1}{(2r+1)(2r+2)} = \log 2.$$

To evaluate the tail exactly, write each summand as $\int_0^1 x^{2r}(1-x)\,dx$. Nonnegativity permits summation under the integral, giving $\int_0^1 (1+x)^{-1}\,dx = \log 2$. Now double the field and move the cutoff from $S = 0$ to $T = 1$. The change is $v = p$. Formula (29) gives

$$E_1[2p] - E_0[p] = -\frac{1}{2} + 2\left(\log 2 - \frac{1}{2}\right) + \left(\log 2 - \frac{1}{2}\right) = 3\log 2 - 2.$$

The three terms are the removed parent shell, the inherited interaction, and the new square. Directly, $E_1[2p] = 4(\log 2 - 1/2)$, which gives the same answer. Every later address is included in both calculations. Here every raw jump has squared magnitude one, so the separate boundary series $\sum_n |\gamma(n)|^2/n$ diverges. Its mixed counterpart cancels that divergence in the ordered energy-jump series. The joined boundary expression is the convergent object.

The removed reading and the two contributions on the common tail appear together in Figure 18. Adding them gives the complete energy change.

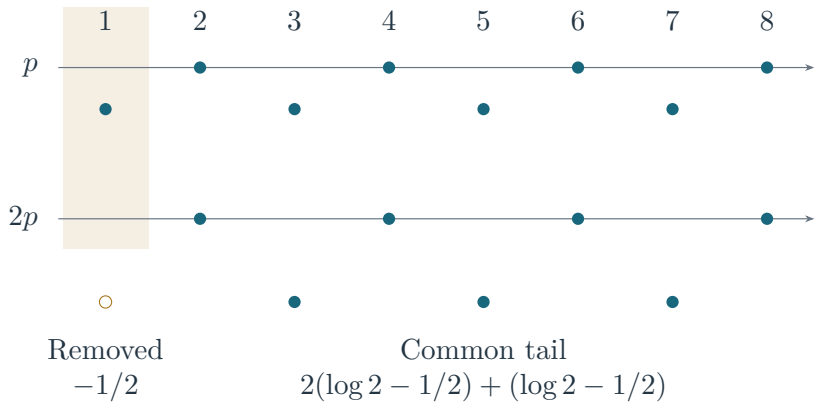



FIGURE 18. Doubling the two-clock field and moving the cutoff removes address one. Every retained odd reading changes from $-1$ to $-2$. The two tail terms are inherited interaction and new square. The pattern continues through all later addresses; its complete change is $3\log 2 - 2$.

10.7. **Weighted tail localization.** The same two-clock example illustrates how a finite calculation can enclose a complete observation. Stop its changed energy at $N = 4$. The removed parent reading contributes $-1/2$, and the only admitted odd address is three. The finite answer is

$$H_4 = -\frac{1}{2} + \frac{3}{3 \cdot 4} = -\frac{1}{4}.$$

Every omitted term lies between zero and $3w_n$. The remaining weight is $1/(N+1)$, giving

$$-\frac{1}{4} \le 3\log 2 - 2 \le -\frac{1}{4} + \frac{3}{5} = \frac{7}{20}.$$

Stopping at eight gives $H_8 = -27/280$ and the interval

$$[-27/280, 199/840].$$

The finite answer may have the wrong sign while its enclosure remains valid. A tail allowance is part of the answer.

This calculation extends to an arbitrary positive observation on the integer addresses. The old and new fields can grow with an outside parameter; their envelopes determine how far the observation must run.

**Theorem 10.4** (Weighted localization)**.** *Let $w_n \ge 0$ be finite, and let $p, q$ be complex fields with finite readings. Fix integers $0 \le S \le T \le N$. Suppose that, for $n > N$,*

$$|p(n)| \le A_n, \qquad |q(n)| \le C_n, \qquad A_n, C_n \ge 0,$$

$$L_N = \sum_{n>N} w_n A_n^2 < \infty, \qquad U_N = \sum_{n>N} w_n C_n^2 < \infty.$$

*Write $v = q - p$ and*

$$H_N = -\sum_{S<n\le T} w_n |p(n)|^2 + \sum_{T<n\le N} w_n \big(2\,\mathrm{Re}(\overline{p(n)}v(n)) + |v(n)|^2\big).$$

*Both complete energies are finite, and their change satisfies*

$$H_N - L_N \le \sum_{n>T} w_n |q(n)|^2 - \sum_{n>S} w_n |p(n)|^2 \le H_N + U_N. \tag{31}$$

*For native weights and constant envelopes $A_n = A$, $C_n = C$, the omitted change lies in $[-A^2/(N+1), C^2/(N+1)]$.*

*Proof.* The finite portions have finitely many finite readings. The two tail energies converge by the stated envelope bounds. Subtracting their absolutely convergent sums gives

$$\sum_{n>T} w_n |q(n)|^2 - \sum_{n>S} w_n |p(n)|^2 = H_N + \sum_{n>N} w_n \big(|q(n)|^2 - |p(n)|^2\big).$$

At each address the last integrand lies between $-|p(n)|^2$ and $|q(n)|^2$. Summing these inequalities proves (31). For native weights, $w_n = 1/n - 1/(n+1)$ telescopes to the stated remaining mass. □

The square of the complete new field gives the upper allowance; the square of the complete old field gives the lower allowance. Expanding the omitted work and bounding its mixed terms separately would usually give a larger enclosure. In the two-clock calculation, the known sign and the identity $|2p|^2 - |p|^2 = 3|p|^2$ improve both sides further.

For a family with envelopes $A_S, C_S$, an error at most $\epsilon$ is guaranteed by

$$N \geq T, \qquad N + 1 \geq \frac{\max(A_S^2, C_S^2)}{\epsilon}.$$

An envelope proportional to $S$ therefore permits localization at a constant multiple of $S^2$. The cutoff grows with the field. No fixed-field quadrature constant is being used as a uniform estimate.

**Problem 22** (A constant field with a moved cutoff)**.** Use native weights, old field $p(n) = 1$, change $v(n) = -1/2$, and cutoffs $S = 1$, $T = 3$. Calculate the complete energy change. Resolve it into the removed shell, inherited interaction and new square. All infinite tails must be evaluated.

**Problem 23** (Choosing a sufficient truncation)**.** Suppose $|p(n)| \leq 2$ and $|q(n)| \leq 3$ on the omitted native tail, and the new activation cutoff is four. Find the least integer $N$ for which the envelope bound guarantees absolute tail error at most $1/10$. If the finite head at that cutoff is $-1/10$, which sign conclusion is justified? Repeat the cutoff calculation for envelopes $2S, 3S$ and activation $2S$, where $S \geq 1$ is an integer.

## 11. Growing signals in weighted windows

Consider a receiver whose observation window moves farther along a sequence while a rapidly alternating component becomes larger. Each sample has a specified mass. The question is whether the observed energy has a stable limit, and whether the oscillation's interaction with the background survives.

The observation in this model has its own weights, specified below; they are not replacements for the native masses. Fix positive integers $a < b$, a bounded weight $\rho \geq 0$, and real functions $A, B$ on $[a, b]$. At scale $N$ observe the addresses $aN < n \leq bN$ with masses

$$v_{N,n} = N^{-2}\rho(n/N).$$

The total mass is $O(N^{-1})$. Let $u(n)$ be a real carrier signal and set the actual readings

$$f_N(n) = B(n/N) + \sqrt{N}\, A(n/N)u(n).$$

The carrier term has growing amplitude. Its square is multiplied by an observation whose mass shrinks at the compensating rate.

11.1. **Four-sample window calculation.** Take $a = 1$, $b = 2$, $\rho(x) = x^{-2}$ and $A = B = 1$, with $u(n) = (-1)^n$. The masses are $v_{N,n} = 1/n^2$. At $N = 4$ the four addresses are $5, 6, 7, 8$, their readings are $-1, 3, -1, 3$, and

$$E_4 = \frac{1}{25} + \frac{9}{36} + \frac{1}{49} + \frac{9}{64}.$$

The result is $35361/78400$. A continuous-address version holds each reading on $[n, n+1)$, so these are its actual right-hand readings. Write $U_N = \sum_{N<n\le 2N} n^{-2}$ and $V_N = \sum_{N<n\le 2N} (-1)^n n^{-2}$. Then exactly

$$E_N = (N+1)U_N + 2\sqrt{N}\, V_N. \tag{32}$$

The first part contains the background square and the carrier square. The second retains their signed interaction.

At $N = 9$ the window becomes $10, \dots, 18$ and the readings alternate between $4$ and $-2$. To compare these observations, extend the formula $f_N(n) = 1 + \sqrt{N}(-1)^n$ to all positive addresses. The change field is $f_9 - f_4 = (-1)^n$. Formula (3) gives

$$E_9 - E_4 = -E_4 + \sum_{n=10}^{18} \frac{f_4(n)^2}{n^2} + 2\sum_{n=10}^{18} \frac{f_4(n)(-1)^n}{n^2} + \sum_{n=10}^{18} \frac{1}{n^2}.$$

The removed old window, admitted old readings, mixed term and new square all appear. The windows happen to be disjoint here; the general identity also handles overlap and changes of weight on retained addresses.

The two observations are drawn in Figure 19. The alternating shape persists, but its amplitude, window and sample masses all change.

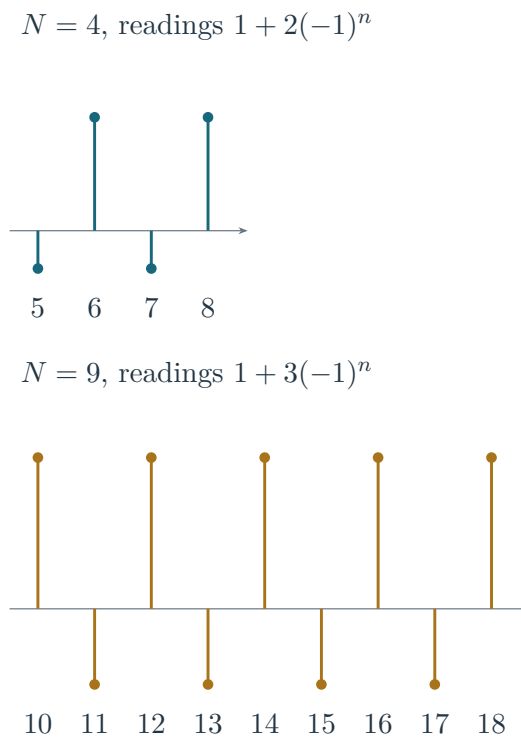



FIGURE 19. The window moves while the alternating amplitude grows. Stems show the actual readings at the two scales, each with mass $1/n^2$. The amplitude, admitted addresses and observation mass all enter the limit; their effects are kept separate in the energy expansion.

The estimate below combines classical summation by parts with bounded-variation quadrature. The finite summation method is developed in [6, Chapter 2]. Its role here is to expose every constant for this moving observation, so the growing amplitude can be handled without a fixed-field assumption.

11.2. **Uniform bound for growing amplitude.** The slowly varying envelopes also need a quantitative description. Total variation adds the magnitudes of successive changes along a partition and takes the supremum over all partitions. A monotone function has variation equal to its endpoint difference in magnitude; for example, $x^{-2}$ on $[1, 2]$ has variation $3/4$. Oscillation and jumps both consume

variation. This is the quantity that the primitive estimate charges to the partner of the rapidly alternating carrier.

For a specified function $h$ of bounded variation on $[a, b]$, put

$$\|h\|_{\mathrm{BV},*} = |h(b)| + \mathrm{Var}_{[a,b]}\, h.$$

Variation includes the assigned point readings. This convention allows step envelopes and exceptional endpoint values. Suppose the three products

$$h_0 = \rho B^2, \qquad h_1 = \rho AB, \qquad h_2 = \rho A^2$$

are of bounded variation. Their Lebesgue integrals are well defined. They correspond to the background square, the mixed product and the carrier square. The constants $P$ and $R$ below control the running sums of the carrier and of its squared fluctuation about $c$. For the alternating carrier, the first running sum stays small and the squared fluctuation vanishes identically. These two controls serve different terms in the energy expansion.

**Theorem 11.1** (Growing carrier under nonuniform observation)**.** *Suppose there are constants $P, R < \infty$ and $c \geq 0$ such that for every finite integer interval $I$,*

$$\left|\sum_{n\in I} u(n)\right| \leq P, \qquad \left|\sum_{n\in I} (u(n)^2 - c)\right| \leq R.$$

*Let $E_N = \sum_{aN<n\leq bN} v_{N,n} f_N(n)^2$. Then*

$$\left|E_N - c\int_a^b h_2(x)\,dx - \frac{1}{N}\int_a^b h_0(x)\,dx\right| \leq \frac{c\,\mathrm{Var}(h_2) + R\|h_2\|_{\mathrm{BV},*}}{N} + \frac{2P\|h_1\|_{\mathrm{BV},*}}{N^{3/2}} + \frac{\mathrm{Var}(h_0)}{N^2}. \tag{33}$$

*In particular the energies converge to $c\int_a^b \rho A^2$. The constants are independent of $N$.*

*Proof.* Expanding the square gives

$$E_N = \frac{1}{N^2}\sum h_0(n/N) + \frac{2}{N^{3/2}}\sum u(n) h_1(n/N) + \frac{1}{N}\sum u(n)^2 h_2(n/N),$$

with every sum over $aN < n \le bN$. On a cell $[(n-1)/N, n/N]$, the difference between the right-hand reading $h(n/N)$ and $h(x)$ is bounded by the variation of $h$ on that cell. Integration and addition give

$$\left| \frac{1}{N} \sum_{aN<n\le bN} h(n/N) - \int_a^b h(x)\,dx \right| \le \frac{\mathrm{Var}(h)}{N}.$$

This also holds with specified exceptional point readings, since their changes contribute to the variation.

Anchor the primitive of $u$ at $aN$. Formula (10) and the first interval hypothesis yield $|\sum u(n)h_1(n/N)| \le P\|h_1\|_{\mathrm{BV},*}$. Apply the same argument to $u^2 - c$ to obtain $|\sum (u(n)^2 - c)h_2(n/N)| \le R\|h_2\|_{\mathrm{BV},*}$. The two quadrature bounds for $h_0, h_2$, together with these two primitive bounds, give (33). □

The alternating carrier has $P = 1$, $c = 1$, $R = 0$. For the four-sample example all three products equal $x^{-2}$ on $[1, 2]$. Their variation is $3/4$, their $\|\cdot\|_{\mathrm{BV},*}$ is one, and their integral is $1/2$. Thus every positive integer $N$ satisfies

$$\left| E_N - \frac{1}{2} - \frac{1}{2N} \right| \le \frac{3}{4N} + \frac{2}{N^{3/2}} + \frac{3}{4N^2}. \tag{34}$$

The field grows as $\sqrt{N}$ while its observed energy tends to $1/2$. Its mixed interaction tends to zero at the explicitly controlled rate $N^{-3/2}$. This is a bound for the growing family itself.

Every bounded periodic carrier with zero mean satisfies the first primitive hypothesis. If $c$ is its period mean square, then $u^2 - c$ also has zero mean and a bounded primitive. Discontinuous envelopes are allowed as long as the three displayed products have bounded variation. The observation, carrier cancellation and envelope variation have separate roles in the estimate.

**Problem 24** (The amplitude threshold)**.** Replace $\sqrt{N}$ by $N^{\alpha}$ in the alternating example. For which $\alpha$ does the energy vanish, remain finite and nonzero, or grow without bound?

**Problem 25** (A carrier with a three-step period)**.** Let $u(3k+1) = 1$, $u(3k+2) = -1$, $u(3k+3) = 0$. In the moving-window model take $a = 1$, $b = 2$ and $\rho = A = B = 1$. Find admissible constants $P, R, c$ from one period and derive an explicit estimate for $E_N$. State the limit and calculate $E_4$ directly.

## 12. Fixed and growing fields

The opening ramp already gave two different limits. Refining its grid with the field fixed sent the defect to zero. Multiplying its amplitude by $\sqrt{q}$ made the defect tend to one. The notation $O_f(q^{-1})$ below means a bound $C_f/q$ with a constant allowed to depend on the entire field. To insert a changing field $f_q$, we must first determine how $C_{f_q}$ grows. The subscript records a substantive dependence.

**Corollary 12.1** (A fixed field)**.** *For every fixed admissible field $f$,*

$$\sup_{0\le\theta<1} |D_{q,\theta}[f]| = O_f(q^{-1}).$$

*If $h = |f|^2$ has no value jumps and its assigned values agree with its continuous traces, the bound improves to $O_f(q^{-2})$.*

*Proof.* There are finitely many boundaries. Their point corrections, value jumps and derivative jumps are fixed, and the regular curvature has finite integral. Bound the point sum by $q^{-1}\sum_b |a_b(h)|$ and use Corollary 5.3 with $c = 0$. Under the stronger hypotheses the point and value-jump terms vanish, leaving only terms of order $q^{-2}$. □

Thus a nonzero persistent defect requires a changing family within these hypotheses. The relevant growth may occur in collision amplitude, the number of boundaries, or the remaining energy data. For example, a sampled collision of $m_q$ equal unit clocks contributes $m_q^2/q$. Its natural constant scale is $m_q$ comparable to $\sqrt{q}$. The carry family has $m_q = p - 1$ and $q = p^2$ at the origin. The pulse family obtains a constant scale through amplitudes comparable to $\sqrt{q}$ at a fixed number of endpoints.

For a growing periodic family, write the energy decomposition as

$$D_{q,\theta}[f] = P_{q,\theta}(f) + R_{q,\theta}(f),$$

where $R$ consists of the last three terms of (4). A limit for the point correction becomes a limit for the whole defect precisely when

the remaining signed contribution vanishes. Corollary 5.3 gives one sufficient route; the carry family requires a more detailed arithmetic estimate.

Refinement does not imply monotonicity of the defect. A decomposition into components must retain the mixed form $K$. Either description requires the complete boundary readings and a bound on the signed remainder for the growing family.

12.1. **Absolute bounds and signed cancellation.** For a general field, the absolute estimate in Corollary 5.3 is the first test. If its right side tends to zero, the point correction determines the limit. If it does not, retain the signed boundary sums before taking absolute values. The failure of that sufficient test does not imply that the residual stays large.

For clocks, constant regular curvature removes the integral term completely. The remaining work is a finite sum. We must estimate the correlation between the background and the reset, weighted by $\psi$, and the correlation between the slope and the reset, weighted by $\widetilde{B}_2$. An exact evaluation of the latter often reduces the question to the former, as the carry applications below demonstrate.

For a sum, the complete field and its interaction matrix retain the same cross terms. For a transformed grid, apply the gcd and offset rule before counting sampled boundaries.

12.2. **Growing pulses with moving overlap.** Two channels can each have zero sampling defect while their difference has a persistent defect. The following complete calculation joins the point conventions, growing amplitude and mixed energy developed above. It uses only interval counting.

Let $q$ be an even positive integer, put $\delta_q = 1/(2q)$, and define half-open gates on the periodic interval by

$$I = [0, 1/2), \qquad J_q = [\delta_q, 1/2 + \delta_q), \qquad A_q = \sqrt{q}\,\mathbf{1}_I, \quad B_q = \sqrt{q}\,\mathbf{1}_{J_q}.$$

The observed signal is $F_q = A_q - B_q$. One gate is delayed by half a grid cell. Determine the energy seen by the integral and by $X_{q,\theta}$, including the offset $\theta = 1/2$ itself.

Combined field. The two channels cancel on their overlap. Consequently

$$F_q = \sqrt{q}\,\mathbf{1}_{[0,\delta_q)} - \sqrt{q}\,\mathbf{1}_{[1/2,1/2+\delta_q)}.$$

Its square is $q$ on two intervals of length $1/(2q)$ and zero elsewhere. Thus its continuous energy is one for every even $q$.

For $0 \le \theta < 1/2$, precisely the samples with indices 0 and $q/2$ lie in these two intervals. The sampled energy is two. For $1/2 \le \theta < 1$, both intervals are missed and the sampled energy is zero. At equality the samples lie at the excluded right endpoints. The assigned half-open readings therefore give

$$D_{q,\theta}[F_q] = \begin{cases} -1, & 0 \le \theta < 1/2, \\ 1, & 1/2 \le \theta < 1. \end{cases} \tag{35}$$

There is no asymptotic remainder in this answer. In particular the two constant limits persist as $q$ grows through the even integers.

At $q = 4$, the delay is $1/8$ and the nonzero amplitudes are $2, -2$. The picture shows how subtracting the overlapping gates leaves two short pieces, then places both grids beneath them.

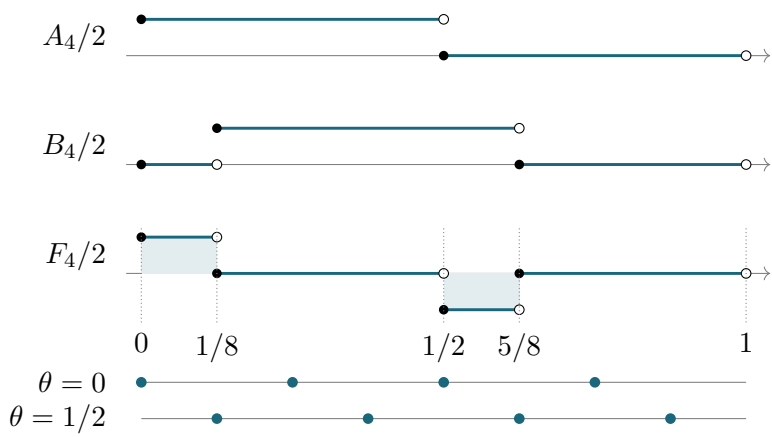



FIGURE 20. Subtracting two delayed gates leaves two short pieces. Filled endpoints are included; open endpoints are excluded. The lower rows place the unshifted and half-cell grids against those pieces. All three fields are divided by two to display their shape.

The curves are divided by two to show their shapes. Filled endpoints are included and open endpoints are excluded. The bottom two rows show sample locations. The unshifted grid meets the two nonzero pieces at their included left endpoints; the half-cell grid meets their excluded right endpoints.

The full calculation can be inspected in four readings.

| Offset | Readings of $F_4$ | Sampled energy | Defect |
|---|---|---|---|
| 0 | $(2, 0, -2, 0)$ | 2 | $-1$ |
| $1/2$ | $(0, 0, 0, 0)$ | 0 | 1 |

Mixed-energy calculation. Each gate has length $1/2$ and contains exactly $q/2$ grid samples, for every offset. Hence each channel has continuous and sampled energy $q/2$, and $D_{q,\theta}[A_q] = D_{q,\theta}[B_q] = 0$. The continuous mixed energy is

$$\int_0^1 A_q B_q \, dx = q(1/2 - \delta_q) = q/2 - 1/2.$$

Their sampled mixed energy is the number of grid points in $[\delta_q, 1/2)$. That number is $q/2 - 1$ when $\theta < 1/2$ and $q/2$ when $\theta \geq 1/2$. Therefore

$$K_{q,\theta}(A_q, B_q) = \begin{cases} 1/2, & 0 \leq \theta < 1/2, \\ -1/2, & 1/2 \leq \theta < 1, \end{cases}$$

$$D_{q,\theta}[F_q] = -2K_{q,\theta}(A_q, B_q).$$

This recovers (35). Adding the two separate channel defects would give zero and lose the entire answer.

The two short intervals support the surviving field. Their endpoints determine which grids read its energy.

12.3. **Further weighted observations.** The problems below transfer the calculation to new observations. Name the object being estimated before choosing a bound.

**Problem 26** (Choose the first operation)**.** For each task, identify the first identity or bound to use and the data it requires. An isolated sample reading changes; an old periodic carrier is paired with a slowly varying weighted partner; a finite energy change has been calculated but its infinite tail remains; a fixed-field error bound is

proposed for a sequence whose amplitude grows. In the last case, state what must be checked before the bound can be used uniformly.

**Problem 27** (Change the two channel amplitudes)**.** In Section 12.2, replace $F_q$ by $aA_q + bB_q$, where $a, b$ are fixed real numbers. Find its continuous energy, sampled energy and defect for both offset ranges. Determine which choices make the defect zero for every offset. Recover the difference example and also calculate the sum of the channels.

**Problem 28** (Growing amplitude on an infinite tail)**.** Fix $a \in \mathbb{R}$ and, for integers $N \geq 1$, put $p_N(n) = a + \sqrt{N}(-1)^n$. Observe $n > N$ with the native weights and write $\mathcal{E}_N = E_N[p_N]$. Separate the two squares and mixed term, bound the complete alternating tail, and find the limit of $\mathcal{E}_N$ with an explicit error allowance. For $a = 1$, choose an integer cutoff $M_N \geq N$ that guarantees the omitted energy is at most $1/N$ using only the uniform envelope for $p_N$. Explain how the cutoff must respond to the growing amplitude.

Fixed fields, growing fields, moving windows and complete tails now have worked examples. The next part organizes their readings through congruences, repetitions and divisor coordinates.

Part 3

# Finite arithmetic tools

The preceding examples specified fields directly. Arithmetic can also specify a field through a digit rule, a congruence or a divisor sum. This part develops coordinates that make those rules available to an energy calculation.

Counting repeating tails first gives a finite reading of ratios built from prime factors. The alignment deficit recovers familiar musical intervals with known finite corrections. The same count can be written as a binary field's energy under two observations.

Congruences organize larger fields by recording where their components meet. The observed intersections form a matrix. Moving an interval edge changes which rows contribute; changing a phase modulus changes the coordinates in which those rows are evaluated.

Common dilation produces another kind of repetition. Primitive shells group repeated pairs, and logarithmic weights connect their prefixes to prime counting. Divisor inversion also recovers the clock sources behind a reset record.

These changes of coordinates preserve a finite calculation. Their use in an estimate requires another decision. We must control the prescribed source and the cost of recovering its reading. The final section of this part examines that cost, preparing the measured reconstruction methods that follow.

## 13. Alignment deficits and the prime lattice

The octave doubles a frequency. A perfect fifth multiplies it by 3/2, and a major third by 5/4. These familiar intervals carry the primes two, three and five. Multiplication combines their prime exponents; division reverses a step. The resulting lattice is a classical description of just intonation.

Repeating digits give this lattice a finite arithmetic reading. Some fractions terminate in a chosen base. Others eventually agree with a reference fraction at every common digit position. Counting both classes gives an alignment proportion. Its shortfall from a limiting proportion is the *alignment deficit*. Ratios of those deficits recover the musical intervals under common scaling, while an exact formula retains the discrepancy at each finite scale [13].

The lattice specifies which ratios are available. The deficit gives each lattice point a finite reading. Its correction measures how far

that reading lies from the limiting ratio. A binary field will express the same count as an energy comparison.

13.1. **A finite count of aligned tails.** Fix a base $b \geq 2$. An integer $m \geq 1$ is *$b$-supported* when every prime factor of $m$ divides $b$. Equivalently, $m$ divides some power of $b$. This condition concerns the primes present in the base, rather than an upper bound on their size.

Let $p \geq 3$ be a prime divisor of $b - 1$. Among the numerators $1 \leq k \leq pm - 1$, count $k/(pm)$ as aligned when its standard base-$b$ expansion terminates or eventually agrees with that of $1/(pm)$ at every sufficiently large common digit position. Terminating expansions use a tail of zeros. The terminating class is included by convention; the nonterminating class requires agreement at the same positions, without a cyclic shift.

For example, in base twelve take $p = 11$ and $m = 2$. There are twenty-one numerators. The numerator eleven gives the terminating fraction $1/2$. The numerators one and twelve have the same eventual tail as $1/22$. Exactly three fractions are aligned, giving proportion $3/21 = 1/7$.

**Proposition 13.1** (Alignment count and deficit)**.** *For $b$-supported $m$ and prime $p \geq 3$ dividing $b - 1$, the alignment proportion and its deficit are*

$$\alpha_b(pm) = \frac{2m - 1}{pm - 1}, \qquad \delta_{b,p}(m) = \frac{2}{p} - \alpha_b(pm) = \frac{p - 2}{p(pm - 1)}.$$

*The aligned numerators are precisely the two classes $k \equiv 0, 1 \pmod{p}$ in $1 \leq k \leq pm - 1$.*

*Proof.* Write $b = hp + 1$. For each $1 \leq r < p$, long division gives $br = (hr)p + r$, so $r/p$ has the single repeating digit $hr$. Distinct nonzero residues give distinct repeating digits.

Choose $L$ with $m \mid b^L$ and put $c = b^L/m$. Moving the base point $L$ places changes $k/(pm)$ to $ck/p$. Since $p$ does not divide $b$, the integer $c$ is invertible modulo $p$. The nonterminating tail agrees with that of the reference exactly when $ck \equiv c \pmod{p}$, or $k \equiv 1 \pmod{p}$. There are $m$ such numerators. The $m - 1$ nonzero multiples of $p$ terminate, because their reduced denominators divide a power of $b$. The other numerators retain the denominator factor $p$ and do

not terminate. Thus $2m-1$ of the $pm-1$ numerators are aligned. Subtracting their proportion from $2/p$ gives the displayed deficit. □

As $m$ grows through supported integers, the alignment proportion tends to $2/p$. The deficit is positive and decreases with $m$. At the base-twelve example it is $2/11 - 1/7 = 3/77$.

13.2. **Prime coordinates and musical intervals.** If the primes dividing $b$ are $q_1, \dots, q_s$, every supported integer has a unique expression

$$m = \prod_{i=1}^{s} q_i^{e_i}, \qquad e_i \in \mathbb{N}_0.$$

The exponent vector is a lattice address. Multiplication by $q_i$ moves one step along its axis. Ratios of supported integers permit negative exponents and give the whole group $\mathbb{Z}^s$.

In base twelve the two axes are two and three. The octave and fifth are therefore present, whereas a major third $5/4$ requires an axis for the prime five. Base thirty supplies all three primes $2, 3, 5$. In those coordinates the octave, fifth and major third are respectively

$$(1,0,0), \qquad (-1,1,0), \qquad (-2,0,1).$$

The exponent of three belongs to the fifth; the major third uses five. Figure 21 shows the two-axis lattice and three intervals measured from the same supported reference.

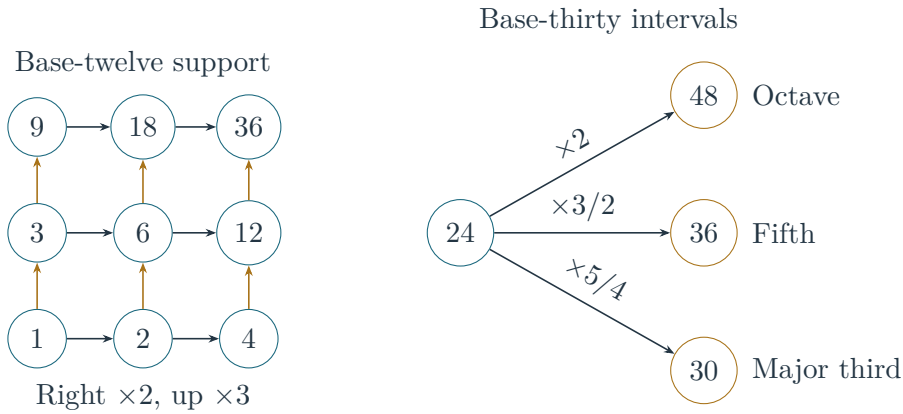



FIGURE 21. Supported integers form a prime-exponent lattice. The left panel labels lattice points by their integer values, not by their deficits. The right panel takes ratios from one reference. The major third uses the prime five, which base thirty adds to the two-axis support of base twelve. The deficit map assigns a finite digit-count coordinate to each supported integer.

The prime-exponent lattice is classical. The alignment deficit places a particular numerical coordinate on its nonnegative cone, obtained from the digit count above. It is not an exactly multiplicative coordinate at finite scale. Its correction is part of the construction.

**Proposition 13.2** (Deficit ratios at finite scale)**.** *Fix $b$ and $p$ as above, abbreviating $\delta_{b,p}$ to $\delta_p$. For supported integers $u, v, t \geq 1$,*

$$\begin{aligned}\frac{\delta_p(tu)}{\delta_p(tv)} &= \frac{ptv-1}{ptu-1} \\ &= \frac{v}{u} + \frac{v-u}{u(ptu-1)}.\end{aligned}$$

*Consequently this ratio tends to $v/u$ as the supported common scale $t$ tends to infinity. Its relative error is*

$$\frac{\delta_p(tu)/\delta_p(tv)}{v/u} - 1 = \frac{v-u}{v(ptu-1)}.$$

*Proof.* Substitute the deficit formula and cancel its common numerator. Subtracting $v/u$ gives numerator $u(ptv-1)-v(ptu-1) = v-u$.

Division by $v/u$ gives the relative error. With $u, v, p$ fixed, both corrections tend to zero. □

At one generator step $m \mapsto qm$, the same formula reads

$$\frac{\delta_p(m)}{\delta_p(qm)} = q + \frac{q-1}{pm-1}.$$

The lattice step is multiplication by $q$. The deficit reading of that step approaches $q$ with the displayed finite excess. Successive ratios telescope exactly along any lattice path with the same endpoints; replacing each ratio by its limiting generator discards these corrections.

For base thirty, use $p = 29$ and reference $u = 24$. The points $v = 30, 36, 48$ give a major third, fifth and octave. At scale $t = 1$, the deficit ratios are

$$\frac{869}{695}, \qquad \frac{1043}{695}, \qquad \frac{1391}{695},$$

approaching $5/4, 3/2, 2$ under common scaling. Their respective relative errors are $1/3475$, $1/2085$, and $1/1390$. The errors shrink under refinement, but remain known at every stage (Figure 22).

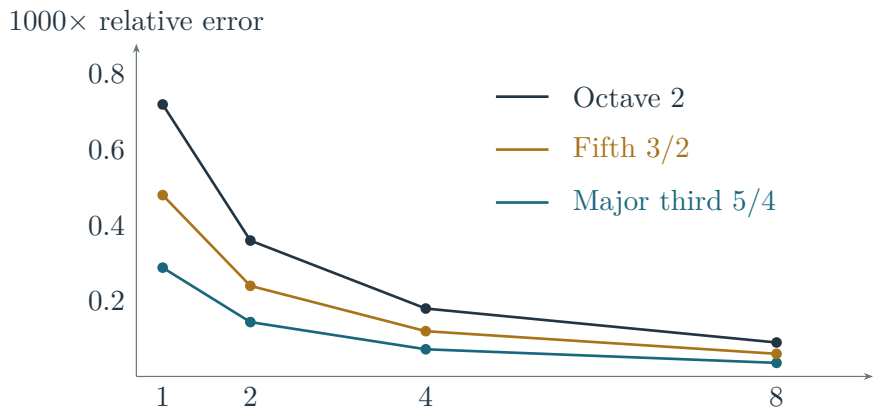



FIGURE 22. Finite deficit ratios approaching the major third, fifth and octave in base thirty. Points show the supported common scales $t = 1, 2, 4, 8$; segments guide the eye. Each error is calculated from the exact formula, with reference 24 and alignment prime 29. No limiting approximation is used for the plotted values.

The full just major scale is obtained from the supported points

$$24, 27, 30, 32, 36, 40, 45, 48$$

relative to twenty-four. The lattice also contains other points in that interval, including twenty-five. The chosen musical scale is a subset of the available arithmetic ratios. Bases thirty and sixty have the same prime support and hence the same lattice; their choices $p = 29$ and $p = 59$ give different finite deficit readings.

13.3. **The deficit as an energy comparison.** The alignment indicator is a field with readings zero and one. Squaring leaves it unchanged, so its alignment proportion is also its mean-square energy. This places the count within the observation language of Section 3.

There is a continuous realization with exactly the same deficit. Partition $[0, 1)$ into $pm$ half-open cells and define the periodic field

$$A_m(x) = \mathbf{1}_{\{\lfloor pmx \rfloor \equiv 0 \text{ or } 1 \pmod p\}} \qquad (0 \le x < 1).$$

Use the right-hand value at every cell boundary. In each group of $p$ cells, precisely two have reading one. Therefore

$$\int_0^1 A_m(x)^2 \, dx = \frac{2}{p}.$$

Observe the actual numerators by the measure

$$\sigma_m = \frac{1}{pm - 1} \sum_{k=1}^{pm-1} \delta_{k/(pm)}.$$

This observation omits the address zero and gives the remaining addresses equal masses adding to one. The alignment count gives

$$E_{\sigma_m}[A_m] = \frac{2m - 1}{pm - 1}, \qquad \int_0^1 A_m^2 \, dx - E_{\sigma_m}[A_m] = \delta_{b,p}(m).$$

Thus the alignment deficit is literally an energy difference for this binary field and these two observations. Figure 23 shows the base-twelve example.

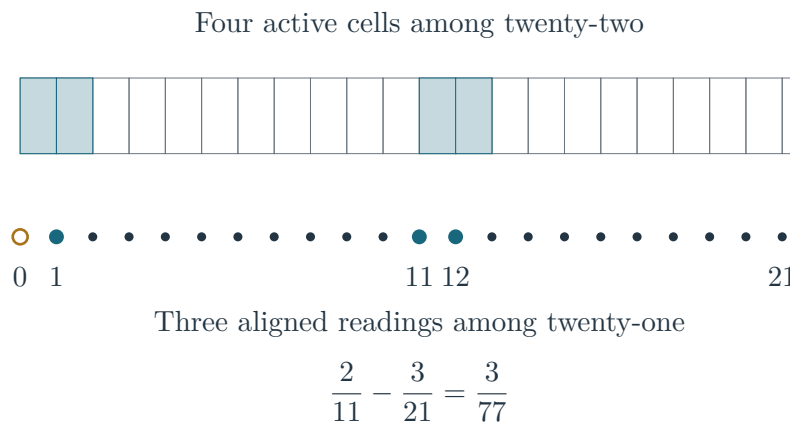



FIGURE 23. The alignment count as an energy comparison at $p = 11, m = 2$. Blue cells have field value one and the other cells zero, giving continuous energy $4/22$. The observation below omits address zero and gives each remaining point mass $1/21$. Its three unit readings give energy $3/21$. The open gold circle marks the omitted address.

The full $pm$-point grid, including zero and using masses $1/(pm)$, would give energy $2/p$ with no defect. Removing zero loses one unit reading and also renormalizes every remaining mass. Both changes produce the deficit. This is an observation change, governed by the general identity in Section 3; it is not the equally weighted full-grid quantity $D_{pm,\theta}$.

Both the alignment deficit and the carry sampling defect compare energies under two observations. Their fields, weights and scales differ. The explicit $1/m$ alignment decay therefore supplies no rate for the growing carry field. Here the correction is known completely, so we can read the multiplicative structure through it at every finite scale.

**Problem 29** (A fifth at finite scale)**.** In base twelve take $p = 11$, $u = 2$ and $v = 3$. Compute the deficit ratio at $t = 1$ and its absolute and relative deviations from $3/2$. Give the same expressions at supported scale $t$. Explain why the ratio approaches the fifth from above.

**Problem 30** (Removing an aligned reading)**.** For the field $A_m$, begin with all $pm$ grid addresses and equal masses $1/(pm)$. Remove the address zero and normalize the remaining masses. Compute the energy loss in two stages, first without renormalization and then with it. Recover the deficit formula.

The next register keeps the same distinction between an arithmetic incidence and the observation that reads it. Several incidences can meet at one row. Their signed amplitudes then have to be combined before the energy is formed.

## 14. Signed collision registers

A congruence records where two arithmetic components can meet. An observation records which of those meetings are actually read. Keeping these two pieces of data together turns a collision count into a finite field to which the energy rules already apply.

Consider the two conditions

$$r \equiv 4 \pmod 7, \qquad r \equiv 1 \pmod 8.$$

They have the common residue 25 modulo 56. An observation on $3 < r < 6$ sees no common reading; one on $24 < r < 27$ sees exactly one. Compatibility of the congruences is therefore only the first part of an interaction calculation. The interval and its weights determine whether that interaction contributes.

The shared residue in Figure 24 contributes only when it lies inside the observed interval.

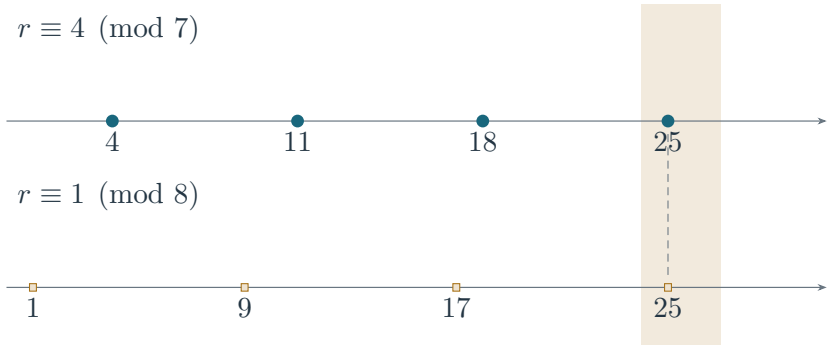



FIGURE 24. Congruence compatibility supplies a common class. The observation must still meet it. This band contains the shared row 25 and the empty row 26. With signed amplitudes $2, -3$ and unit masses, the observed energy is $(2-3)^2 = 1$.

14.1. **Incidences and the observation kernel.** Let $\mathcal{A}$ be a finite set of arithmetic components. A component $\xi$ has modulus $c_\xi \geq 1$, residue $\rho_\xi$, real amplitude $z_\xi$ and orientation $\sigma_\xi \in \{1, -1\}$. Define its incidence and the combined row by

$$I_\xi(r) = \mathbf{1}_{r \equiv \rho_\xi \pmod{c_\xi}}, \qquad \Delta(r) = \sum_{\xi \in \mathcal{A}} \sigma_\xi z_\xi I_\xi(r).$$

Give the integer rows nonnegative weights $v(r)$ of finite support. The energy and intersection kernel are

$$M_v = \sum_r v(r)\Delta(r)^2, \qquad K_v(\xi, \eta) = \sum_r v(r) I_\xi(r) I_\eta(r). \tag{36}$$

Think of each component as a column and each address $r$ as a row. The entry $I_\xi(r)$ records whether that component is present there; its signed amplitude determines its contribution to the reading. Several columns can contribute to the same observed row. In this section $\Delta(r)$ names that combined reading; it is not the backward difference $\Delta h(n)$ used in the weighted identities.

**Proposition 14.1** (Signed incidence formula)**.** *The complete energy is*

$$M_v = \sum_{\xi,\eta\in\mathcal{A}} \sigma_\xi\sigma_\eta z_\xi z_\eta K_v(\xi,\eta). \tag{37}$$

*The sum is over ordered pairs. In particular, both mixed orientations are retained.*

*Proof.* Expand the square defining $M_v$ and interchange the finite sums. The coefficient of $\sigma_\xi\sigma_\eta z_\xi z_\eta$ is exactly the second expression in (36). □

This is the collision-matrix calculation of Section 9, with its entries evaluated by congruences. The matrix $K_v$ is positive semidefinite because it is a Gram matrix of incidence vectors. Its signed quadratic form nevertheless contains negative mixed terms. Replacing those terms by absolute values changes the energy being estimated.

14.2. **Common residues and finite intervals.** The Chinese remainder theorem supplies the common residue [1, Chapter 5]. Parentheses denote the greatest common divisor and brackets the least common multiple. Write

$$g = (c_1, c_2), \qquad \ell = [c_1, c_2].$$

The notation $g \mid d$ means that $g$ divides $d$. A bar denotes an inverse in the displayed modulus. The two incidences meet exactly when $g \mid \rho_2 - \rho_1$. If they meet, solve

$$t \equiv \frac{\rho_2 - \rho_1}{g}\,\overline{(c_1/g)} \pmod{c_2/g},$$
$$R \equiv \rho_1 + c_1 t \pmod{\ell}, \quad 0 \le R < \ell.$$

For $c_2/g = 1$ take $t = 0$. These formulas follow by substituting $r = \rho_1 + c_1 t$ into the second congruence. The reduced moduli are coprime, so the displayed inverse exists. Every common row is $R + j\ell$.

If the observation is supported on $Y < r < 2Y$ and $\ell > Y$, there is at most one common row. Its first possible representative is

$$j_0 = \left\lfloor \frac{Y - R}{\ell} \right\rfloor + 1, \qquad r_0 = R + j_0\ell.$$

The kernel is $v(r_0)$ when compatibility holds and $r_0 < 2Y$; otherwise it is zero. The inequality $r_0 > Y$ is already enforced. For a longer interval, retain the full sum $\sum_{j:\, Y<R+j\ell<2Y} v(R + j\ell)$.

**Problem 31** (A shared row with opposing amplitudes)**.** On $24 < r < 27$ give each integer row weight one. Use the residues 4 modulo 7 and 1 modulo 8, amplitudes $2, 3$, and orientations $+1, -1$. Compute the energy and its four ordered terms.

The opening common residue supplies a complete two-column calculation. On rows $25, 26$ both incidences have column $(1, 0)^T$. With unit masses their Gram is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Signed amplitudes $(2, -3)^T$ give energy one. If the first row has mass two instead, every Gram entry doubles and the energy is two. The residue calculation has not changed; the observation has.

14.3. **A divisibility register.** A useful arithmetic instance fixes an odd prime $p$ and uses moduli $p/2 < c < p$ and columns $1 \le b < p$. Put

$$\Delta[f](r) = \sum_{c} a_c \sum_{1 \le b < p} f_b \big( \mathbf{1}_{c \mid p^2 - br} - \mathbf{1}_{c \mid p^2 + br} \big), \qquad 1 \le r < p, \quad (38)$$

where $a_c, f_b$ are real coefficients. The row weights $v(r)$ still specify the observation; $a_c$ and $f_b$ specify the field. A column $b$ is a unit modulo $c$ when $(b, c) = 1$. Then its inverse $\bar{b}$ satisfies $b\bar{b} \equiv 1 \pmod{c}$. A unit component $(c, b, \sigma)$ has residue $\rho \equiv \sigma p^2 \bar{b} \pmod{c}$. A nonunit $b$ has no incidence: any common divisor of $b, c$ would divide $p^2$, although $c < p$. For two unit components, compatibility becomes

$$(c_1, c_2) \mid \sigma_1 b_2 - \sigma_2 b_1. \quad (39)$$

Multiplying the common-residue condition by the unit $b_1 b_2 \overline{p^2}$ proves this equivalence. The interval test is still needed after (39).

Let $d_c(r)$ denote the $c$ term in (38). If $|a_c| \le A$, $|f_b| \le B$ and $\sum_r v(r) \le V$, then

$$0 \le \sum_{r} v(r) \sum_{c} d_c(r)^2 \le 16 A^2 B^2 p V. \quad (40)$$

For fixed $c, r$ each sign permits at most two columns, since $c > p/2$. Hence $|d_c(r)| \le 4AB$, and summing over fewer than $p$ moduli proves the bound. This retains all pairs at a common modulus, including distinct columns.

The remaining energy is the signed sum over different moduli. A first-moment target may require less than control of that energy:

Cauchy–Schwarz gives

$$\left|\sum_r W(r)\Delta(r)\right|^2 \le \left(\sum_r |W(r)|\right)\left(\sum_r |W(r)|\Delta(r)^2\right).$$

The implication goes from a second-moment bound to a first-moment bound. A row pair $T, -T$ has zero unweighted sum and energy $2T^2$, so the converse need not hold.

**Problem 32** (Compatibility after a sign change)**.** Keep the first residue 4 modulo 7 and replace the second residue 1 modulo 8 by $-1$ modulo 8. Find the common residue modulo 56. Compare the two intersection kernels on $24 < r < 27$.

**Problem 33** (Weighted intersections over several periods)**.** Use components $r \equiv 1 \pmod 3$ and $r \equiv 2 \pmod 4$, with signed amplitudes $2, -1$. Observe the rows $1 \le r \le 25$, giving row ten weight two and every other row weight one. Find the full $2\times2$ incidence Gram matrix and the observed energy. Recompute the energy after changing the second amplitude to $+1$. Explain why an at-most-one-common-row shortcut would fail for this observation.

## 15. Boundary localization and reciprocal transfer

Moving the edge of a register changes only the columns crossed by that edge. The cost can be counted before any smooth approximation is chosen. For example, at $p = 11, r = 4$, changing column $b = 10$ by $h$ changes (38) by

$$h(a_9 - a_7).$$

Indeed $p^2 - br = 81$ and $p^2 + br = 161$ have, respectively, the divisors 9 and 7 in $11/2 < c < 11$. The sign and the two modulus coefficients remain visible in the change.

### 15.1. **Column budgets and energy transfer.**

Let $\tau(n)$ count the positive divisors of $n$.

**Lemma 15.1** (Finite-column transfer)**.** *Suppose $f - g$ is supported on at most $J$ columns and $\|f - g\|_\infty \le B$. For (38), with $|a_c| \le A$,*

*every $1 \le r < p$ satisfies*

$$|\Delta[f](r) - \Delta[g](r)| \le AB \sum_{b:\, f_b \ne g_b} \big(\tau(p^2 - br) + \tau(p^2 + br)\big) \qquad (41)$$
$$\le C_\varepsilon AB J p^{\varepsilon} \qquad (\varepsilon > 0).$$

*The constant is independent of the row and changed columns.*

*Proof.* Each incidence uses a divisor of one of the displayed positive integers. Both integers are less than $2p^2$ because $b, r < p$. Triangle inequality over the changed columns proves the first bound.

For completeness, $\tau(n) \le C_\delta n^\delta$ for every $\delta > 0$. If a prime $\ell \ge 2^{1/\delta}$, then $a + 1 \le 2^a \le \ell^{\delta a}$. For each of the finitely many smaller primes, $\sup_{a \ge 0}(a+1)\ell^{-\delta a}$ is finite. Multiplication over prime powers proves the divisor estimate. Take $\delta = \varepsilon/2$ in the first bound. □

The same change has a controlled effect on energy without dropping the inherited interaction. Set $h(r) = \Delta[f](r) - \Delta[g](r)$ and $M_v[f] = \sum_r v(r)\Delta[f](r)^2$. Then

$$M_v[f] - M_v[g] = 2\sum_r v(r)\Delta[g](r)h(r) + \sum_r v(r)h(r)^2, \quad (42)$$

$$\Big|\sqrt{M_v[f]} - \sqrt{M_v[g]}\Big| \le \|h\|_\infty \sqrt{\sum_r v(r)}.$$

The first formula is the field-change identity. The second is the reverse triangle inequality in the weighted square norm. Squaring a bound for $h$ alone would miss the first term in (42).

15.2. **An adjustable smoothing width.** Smoothing has two costs. A wider transition changes more columns; a narrower transition has larger derivatives. Both costs must refer to the same register.

Choose fixed smooth functions $\chi$ on $[0,1]$ and $u$ on the real line. Let $\chi$ vanish near zero, and let $u(t) = 0$ for $t \le 0$, $u(t) = 1$ for $t \ge 1$. For $1 \le J \le p$ put

$$f_b^\sharp = \chi(b/p), \qquad B^{[J]}(\beta) = \chi(\beta)u\big(p(1-\beta)/J\big).$$

Extend $B^{[J]}$ by zero outside $[0,1]$. The sharp and smooth tables differ only on $p - J < b < p$. Differentiation and a change of variable over an interval of length $J/p$ give

$$\|(B^{[J]})'\|_1 \le C, \qquad \|(B^{[J]})''\|_1 \le Cp/J. \qquad (43)$$

Here and below the profile constants are fixed.

**Proposition 15.2** (Balancing a boundary budget)**.** *Assume that the chosen modulus weights admit the smooth-row estimate*

$$|\Delta[B](r)| \le C_0\sqrt{p}\,\tau(r)\sqrt{\|B'\|_1\|B''\|_1} \tag{44}$$

*for all the profiles $B = B^{[J]}$, uniformly in $p, r, J$. The notation $\Delta[B]$ means (38) with $f_b = B(b/p)$. If the modulus weights are uniformly bounded, then*

$$|\Delta[f^\sharp](r)| \le C_\varepsilon J p^\varepsilon + C_1 \frac{p}{\sqrt{J}}\tau(r). \tag{45}$$

*Choosing $J = \lceil p^{2/3} \rceil$ gives $|\Delta[f^\sharp](r)| \le C_\varepsilon p^{2/3+\varepsilon}$.*

*Proof.* Lemma 15.1 bounds the difference between the sharp and smooth rows by $C_\varepsilon J p^\varepsilon$. Substituting (43) into (44) bounds the smooth row by $C_1 p\tau(r)/\sqrt{J}$. Add the two bounds and use $\tau(r) \le C_\varepsilon p^\varepsilon$. □

Thus this hypothesis supplies normalized row size $O_\varepsilon(p^{-1/3+\varepsilon})$ after division by $p$. The transition width is chosen after its column cost has been estimated. The same balancing step can be used wherever those two costs are known. The smooth-row bound is an explicit additional input; it is not a consequence of the incidence identity. In the motivating short-variable calculation that input uses completion of exponential sums and a classical Weil bound. Its analytic proof is not needed to apply the budget principle to another family with its own smooth estimate.

The exponents also locate the limit of this particular calculation. Writing $J = p^\theta$ gives the two powers $\theta$ and $1-\theta/2$. Their maximum is least at $\theta = 2/3$. Improving the exponent requires a better estimate for at least one cost or cancellation between the retained terms.

The two allowances balance at the crossing in Figure 25. Moving the width in either direction makes the larger allowance worse.

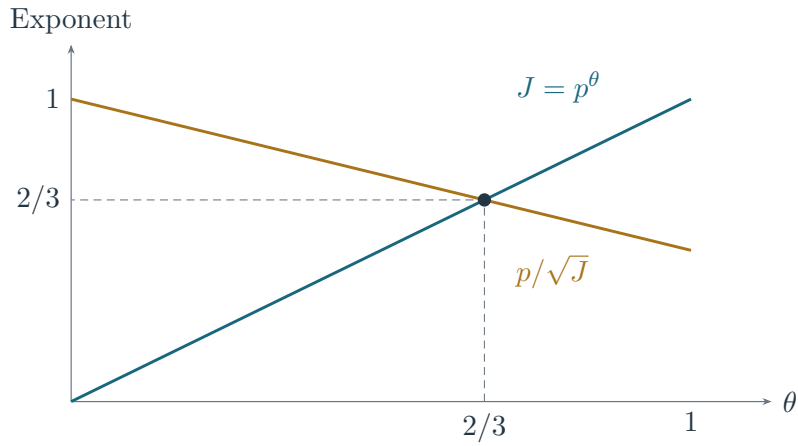



FIGURE 25. Balancing the two stated allowances. Their powers of $p$ are $\theta$ and $1 - \theta/2$. The larger exponent is smallest where the lines meet. This optimization uses the assumed smooth-row bound; it does not supply that analytic input.

15.3. **Reciprocal phase transfer.** A smooth-row calculation may contain an inverse modulo a large modulus. A finite identity transfers that inverse to a smaller one and records the correction. Write $e(x) = \exp(2\pi i x)$.

**Proposition 15.3** (Reciprocal phase identity)**.** *For coprime integers $r, c > 1$ and any integer $A$,*

$$e(A\bar{r}/c) = e(-A\bar{c}/r)\, e(A/(rc)). \tag{46}$$

*The bars denote inverses modulo the adjacent denominators.*

*Proof.* Choose $1 \le u < c$ with $ru \equiv 1 \pmod{c}$ and put $v = (ru - 1)/c$. Then $ru - cv = 1$, so $v \equiv -\bar{c} \pmod{r}$. Dividing by $rc$ gives $u/c = v/r + 1/(rc)$. Exponentiation proves the identity; integer $A$ makes the representatives immaterial. □

For $r = 4, c = 7$, the relation $4 \cdot 2 - 7 \cdot 1 = 1$ gives

$$e(2A/7) = e(-3A/4)e(A/28).$$

At $A = 121$ the two phases are $242/7$ and $-363/4 + 121/28$; they differ by 121, so their exponentials agree. The transferred inverse has modulus four. Its reciprocal correction $e(A/(rc))$ remains part of

the field and may vary rapidly when $A$ grows. The identity changes coordinates; any cancellation estimate must still control the resulting amplitudes and phases.

**Problem 34** (A change of boundary cost)**.** Suppose a related calculation gives $|\Delta^\sharp| \le C(J + pJ^{-1/3})$, with $1 \le J \le p$. Choose the power of $p$ for $J$ that minimizes this allowance. What rate follows for $\Delta^\sharp/p$?

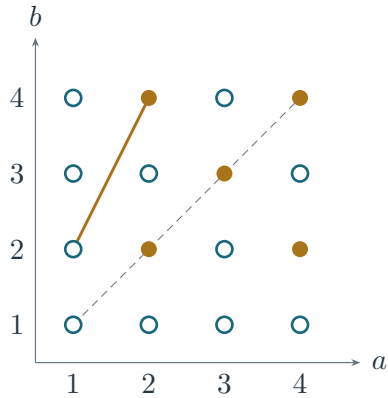
**Problem 35** (The reciprocal phase correction)**.** In the reciprocal phase identity take $r = 5$, $c = 7$ and $A = 3$. Compute both modular inverses and verify the identity by comparing its arguments modulo one. Determine the absolute error caused by omitting the correction $e(A/(rc))$, where $e(x) = \exp(2\pi i x)$. For general coprime $r, c > 1$ and integer $A$, derive an upper allowance for that error in terms of $|A|/(rc)$.

## 16. Primitive shells and logarithmic transforms

Pairs $(1, 2), (2, 4), (3, 6)$ have the same ratio. A kernel unchanged by common dilation assigns them the same reading. In a square cutoff $1 \le a, b \le M$, the primitive pair $(1, 2)$ appears $\lfloor M/2 \rfloor$ times. Counting these repetitions is the starting point of a transform that separates primitive structure from its repeated observations.

In the arithmetic application, the kernel has a fixed modulus $q$, called its *carrier modulus*. The cutoff changes while $q$ stays fixed. This is the arithmetic counterpart of keeping a field fixed while changing its observation. The general decomposition below needs only invariance under common dilation.

At cutoff four, the rays in Figure 26 group each primitive pair with all its admitted repetitions.




Open circles are primitive pairs.

Filled circles repeat a smaller pair.

$(1, 2)$ appears twice at cutoff four.

$$\mathcal{A}(4) = \sum_{m=1}^{4} \lfloor 4/m \rfloor P(m)$$

FIGURE 26. Common dilation groups the square into primitive rays. A primitive pair with largest coordinate $m$ has $\lfloor 4/m \rfloor$ representatives in this square. Every ordered pair is included once in this grouping; the kernel value, including its sign, travels with the ray.

16.1. **Dilation-invariant kernels.** Fix a maximum cutoff $M_0$. Let a real kernel $R(a, b)$ on $1 \le a, b \le M_0$ satisfy

$$R(da, db) = R(a, b) \qquad \text{whenever } d \max(a, b) \le M_0.$$

Define its square prefixes and primitive shells by

$$\mathcal{A}(M) = \sum_{a,b \le M} R(a, b), \qquad P(m) = \sum_{\substack{\max(a,b)=m \\ (a,b)=1}} R(a, b), \qquad \mathcal{A}(0) = 0. \tag{47}$$

The inversion uses the Möbius coefficient $\mu(r)$. For a positive integer $r$, $\mu(r) = 0$ if a prime square divides $r$; otherwise $\mu(r) = (-1)^k$, where $k$ is its number of distinct prime factors, with $\mu(1) = 1$. Thus $\mu(3) = -1$, $\mu(15) = 1$ and $\mu(9) = 0$.

**Proposition 16.1** (Primitive-shell decomposition)**.** *For* $1 \le M \le M_0$,

$$\mathcal{A}(M) = \sum_{m \le M} \lfloor M/m \rfloor \, P(m), \tag{48}$$

$$\mathcal{A}(M) - \mathcal{A}(M-1) = \sum_{m \mid M} P(m),$$

$$P(m) = \sum_{d \mid m} \mu(d)\big(\mathcal{A}(m/d) - \mathcal{A}(m/d-1)\big). \tag{49}$$

*Proof.* Every ordered pair is uniquely $d(a,b)$ with $(a,b)=1$. A primitive pair with maximum $m$ has $\lfloor M/m \rfloor$ admissible multiples. This proves the first identity. The increment of $\lfloor M/m \rfloor$ is one precisely when $m \mid M$, which proves the second. Finally $\sum_{d \mid n} \mu(d)$ is one at $n = 1$ and zero otherwise: for $n > 1$ its squarefree divisors give the expansion of $\prod_{\ell \mid n}(1-1)$. Applying this identity to the second formula gives (49). □

The inversion is classical Möbius inversion [1, Chapter 2]. The useful choice in an application is the kernel whose common dilations really preserve the reading. Signed values are allowed throughout; no positivity or asymptotic hypothesis enters the proposition.

**Problem 36** (Repeated pairs)**.** Take $R(a,b) = 1$ and $M = 4$. Calculate the four primitive-shell coefficients and reconstruct $\mathcal{A}(4)$.

16.2. **The logarithmic prefix operation.** The factor $\lfloor M/m \rfloor$ counts every repetition of a primitive pair. A different weighting of those repetitions can select prime powers. The logarithmic operation below makes that change exactly, using the factorization identity for $\log n$.

Define the prime-power weight and its cumulative sum by

$$\Lambda(n) = \begin{cases} \log \ell, & n = \ell^j,\ \ell \text{ prime},\ j \geq 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \Psi(x) = \sum_{n \leq x} \Lambda(n).$$

Empty sums are zero. Unique factorization gives $\log n = \sum_{d \mid n} \Lambda(d)$, so Möbius inversion gives $\Lambda(n) = -\sum_{d \mid n} \mu(d) \log d$. Summing over $n \leq x$ yields the finite identity

$$\Psi(x) = -\sum_{d \leq x} \mu(d) \log d \, \lfloor x/d \rfloor \,. \tag{50}$$

These are the classical divisor identities for $\Lambda$ [1, Chapter 2].

Apply the same coefficients to the square prefixes:

$$\mathcal{K}(N) = -\sum_{d \leq N} \mu(d) \log d \, \mathcal{A}(\lfloor N/d \rfloor). \tag{51}$$

The calligraphic letter distinguishes this prefix reading from the mixed defect $K_q(f,g)$ used earlier.

**Theorem 16.2** (Logarithmic shell transform)**.** *Under the hypotheses of Proposition 16.1, for every* $1 \le N \le M_0$,

$$\mathcal{K}(N) = \sum_{m \le N} P(m)\Psi(N/m). \tag{52}$$

*Proof.* Insert (48) into (51). The coefficient of $P(m)$ after exchanging the finite sums is

$$-\sum_{d \le N/m} \mu(d) \log d \left\lfloor \frac{N}{dm} \right\rfloor = \Psi(N/m),$$

by (50). The equality $\lfloor \lfloor N/d \rfloor / m \rfloor = \lfloor N/(dm) \rfloor$ ensures that no endpoint changes. □

This theorem composes gcd grouping with classical convolution. It applies to any kernel satisfying the dilation hypothesis. Its usefulness depends on the resulting shell coefficients and the estimates available for the shorter prefixes.

16.3. **A kernel supplied by clock boundaries.** The midpoint sawtooth supplies such a kernel. Recall $\psi$ from Section 4, and put

$$s(h,k) = \sum_{j=1}^{k-1} \psi(j/k)\psi(hj/k), \qquad s(h,1) = 0.$$

Fix an odd prime $p$, let $q = p^2$, and restrict all indices to $1 \le a, b < p$. Define

$$R_q(a,b) = s(a\bar{q}, b) + s(b\bar{q}, a). \tag{53}$$

Every inverse uses the denominator of its own sum. The carrier $q$ stays fixed in $\mathcal{A}_q(M)$, $P_q(m)$ and $\mathcal{K}_q(N)$.

Here is the connection to the boundary calculus. For $f_a(x) = 2\psi(ax)$,

$$K_{q,0}(f_a, f_b) = \frac{1}{q} - \frac{a/b + b/a}{3q^2} + \frac{4}{q} R_q(a,b). \tag{54}$$

To verify it first take $(a,b) = 1$. The product $f_a f_b$ has point correction one at zero and no other sampled boundary. At $j/a$, $1 \le j < a$,

its value jump is $-4\psi(bj/a)$; the other clock gives the symmetric terms. Their boundary sum is

$$\frac{4}{q}\sum_{j=1}^{a-1}\psi(bj/a)\psi(qj/a)+\frac{4}{q}\sum_{j=1}^{b-1}\psi(aj/b)\psi(qj/b)=\frac{4}{q}R_q(a,b),$$

after permuting residues by $q$. The slope-jump contributions, including the origin, are $-2q^{-2}(b/(6a)+a/(6b))$. This uses $\sum_{j=0}^{a-1}\widetilde{B}_2(qj/a)=1/(6a)$. The regular curvature is the constant $8ab$ and has zero Bernoulli integral. Theorem 5.1 now gives (54).

For a common divisor $d=(a,b)$, remove $d$ from both clocks. Integer dilation preserves their continuous pairing and permutes the $q$-grid, since $(d,q)=1$. It also preserves the kernel:

$$s(dh,dk)=s(h,k).$$

Indeed, group the summation index by its residue modulo $k$ and use

$$\sum_{j=0}^{d-1}\psi((x+j)/d)=\psi(x).$$

Inverses of $q$ reduce consistently to smaller denominators. This proves both the general pair formula and the common-dilation hypothesis for $R_q$.

Thus primitive-shell grouping applies to the signed remainder already produced by the boundary decomposition. The classical Dedekind sum provides a compact notation for its boundary phases. Rademacher and Grosswald [23] give the classical theory, using the same midpoint value at integer resets.

16.4. **A finite prime-counting example.** For $m>1$, pair the two orientations of a primitive collision. The unit residues in the second Dedekind term cancel under $k\mapsto m-k$, giving

$$P_q(m)=2\sum_{\substack{1\le k<m\\(k,m)=1}}s(m\bar{q},k). \tag{55}$$

Also $P_q(1)=0$. Directly from the defining sums,

$$s(1,2)=0,\qquad s(1,3)=\frac{1}{18},\qquad s(2,3)=-\frac{1}{18},\qquad s(1,4)=\frac{1}{8}.$$

Because $p^2 \equiv 1 \pmod{12}$ for primes $p \geq 5$, these give

$$P_q(1) = P_q(2) = P_q(3) = 0, \qquad P_q(4) = \frac{1}{9}, \qquad P_q(5) = \frac{5}{36}, \quad (56)$$

with the last coefficient used when $p \geq 7$. For $m = 4$ only $k = 3$ contributes; for $m = 5$ the nonzero terms are $k = 3, 4$.

Take $q = 121$ and $N = 10$. In (52), the first three shells vanish and all $m \geq 6$ have $10/m < 2$, so

$$\mathcal{K}_{121}(10) = \left(\frac{1}{9} + \frac{5}{36}\right) \log 2 = \frac{1}{4} \log 2.$$

This is a complete example of prime counting inside a transformed collision reading.

Primitive means coprime coordinates, not a prime shell index. For instance (55) gives $P_{49}(6) = -2/5$ and $P_{121}(6) = 2/5$, whereas $\Lambda(6) = 0$. The shell keeps its carrier dependence and signs. Any estimate for this identity must retain those features.

**Problem 37** (Logarithmic transform of a constant kernel)**.** For $R(a, b) = 1$, compute $\mathcal{K}(4)$ directly from the prefixes $\mathcal{A}(M) = M^2$ and then from the primitive shells.

**Problem 38** (Signed primitive shells)**.** Let $R(a, b) = 2$ when $a = b$, let $R(a, b) = -1$ when $a = 2b$ or $b = 2a$, and let it be zero otherwise. Verify common-dilation invariance. Find every nonzero primitive-shell coefficient and the prefixes $\mathcal{A}(M)$ for $1 \leq M \leq 6$. Compute $\mathcal{K}(6)$ both from the logarithmic prefix definition and from the shell transform. Explain why some prefix increments vanish although the observed square grows.

## 17. Reset coordinates and primitive energy

Primitive shells separated a repeated arithmetic reading into its divisor contributions. The same inversion can answer a question about the field itself. Given its combined reset amplitudes, which source produces them?

A reset record lists the combined amplitudes that meet at each boundary. The source lists the clocks that produce them. These are different coordinates for the same finite field. Selecting some resets

therefore requires reconstructing a source, even when the selection itself is just a list of zeros and ones.

Use the midpoint clocks $f_h(x) = 2\psi(hx)$ and write $F_c = \sum_{h=1}^{N} c_h f_h$, with real coefficients for the examples. Here $h$ is a frequency on the unit circle. The native clocks $\{u/j\}$ in Section 10 instead use $j$ as a period. Their divisor tables run in opposite directions, so we specify the direction of summation each time.

17.1. **Four clocks and their resets.** At a reduced boundary $a/k$, exactly the frequencies divisible by $k$ reset. Define the *reset mass*

$$m_k(c) = \sum_{\substack{h \le N \\ k \mid h}} c_h.$$

The field jump there is $-2m_k(c)$. Include $k = 1$ to record the common reset. At $N = 4$ the incidence table is

$$m = Zc, \qquad Z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $c = (1, -1, 1, 1)^T$ this gives $m = (2, 0, 1, 1)^T$. The two clocks resetting at one half cancel there. The individual clocks remain in the field and still affect its other readings.

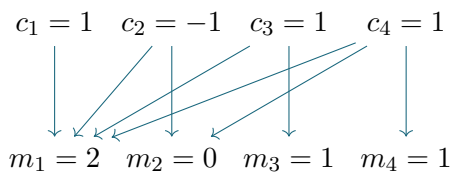



FIGURE 27. Each arrow records a divisibility relation $k \mid h$. Adding the incoming signed coefficients gives the reset mass. At denominator two, the contributions $-1$ and 1 cancel while both clocks remain in the field.

An arrow means that $k \mid h$, so $c_h$ contributes to $m_k$. Read each bottom entry by adding all incoming signed coefficients.

Recover the source starting at the largest denominator. Here $c_4 = m_4$, $c_3 = m_3$, $c_2 = m_2 - m_4$ and $c_1 = m_1 - m_2 - m_3$. Suppose only denominators one and two are retained. The selected

record is $(2,0,0,0)^T$, whose source is $(2,0,0,0)^T$. Deleting coefficients three and four would instead leave $(1,-1,0,0)^T$, with reset record $(0,-1,0,0)^T$. That operation selects a different field.

**Proposition 17.1** (Reconstruction and reset selection)**.** *For any finite source on* $1,\dots,N$*,*

$$c_h = \sum_{j \le N/h} \mu(j) m_{hj}(c).$$

*If* $\sigma_k \in \{0,1\}$ *specifies retained reset denominators, the unique selected source is*

$$P_\sigma c = Z^{-1}\,\mathrm{diag}(\sigma) Z c, \qquad P_\sigma^2 = P_\sigma.$$

*For either continuous or sampled Gram* $G$*, write* $b = P_\sigma c$*,* $r = c - b$*. Its complete energy is*

$$c^* G c = b^* G b + 2\,\mathrm{Re}(b^* G r) + r^* G r.$$

*The selector need not be orthogonal for that observation.*

*Proof.* The coefficient of $c_{h\ell}$ in the proposed inverse is $\sum_{j|\ell} \mu(j)$, equal to one when $\ell = 1$ and zero otherwise. All sums are finite. This proves inversion and uniqueness. Conjugating the diagonal mask gives $P_\sigma$; its square is itself because $\sigma_k^2 = \sigma_k$. Expansion of $c = b + r$ proves the energy identity, with no orthogonality assumption. □

This is finite Möbius inversion applied to the reset data. For the four-clock selection above, the continuous energies of the whole field, selected field and remainder are 1, 4/3 and 8/9. Their mixed contribution is $-11/9$. A selected field can have more energy than the whole field because its interaction with the remainder was negative.

Figure 28 keeps the reconstructed source beside the selected reset record. The negative mixed contribution explains why separating the field increases the sum of the individual energies.

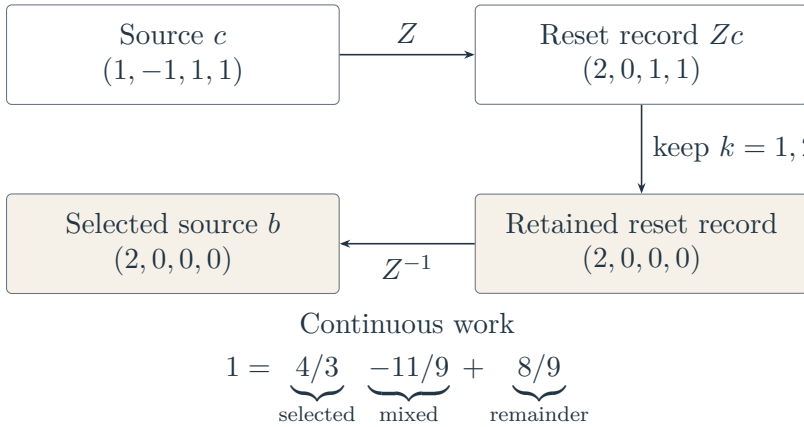



FIGURE 28. Select resets in the coordinates that record them, then reconstruct the source. The selected field has more continuous energy than the whole field in this example. Its negative interaction with the remainder is part of the original energy.

17.2. **Coordinates that separate continuous work.** Reset masses describe jumps. A weighted divisor sum instead separates the continuous energy. Put

$$q_d(c) = \sum_{\substack{h \le N \\ d \mid h}} \frac{c_h}{h}, \qquad J_2(d) = \sum_{e \mid d} \mu(d/e) e^2 = d^2 \prod_{\ell \mid d,\ \ell \text{ prime}} (1 - \ell^{-2}).$$

The product is positive, with $J_2(1) = 1$. The numbers $q_d$ are not the reset masses $m_d$. They include the frequency denominator.

**Proposition 17.2** (Primitive energy coordinates)**.** *The continuous work of $F_c$ is*

$$\int_0^1 |F_c|^2 \, dx = \frac{1}{3} \sum_{d \le N} J_2(d) |q_d(c)|^2. \tag{57}$$

*Define*

$$g_n = \sum_{h \mid n} h \mu(n/h) f_h.$$

*Then $F_c = \sum_{n \le N} q_n(c) g_n$ and $\int_0^1 \overline{g_n} g_m \, dx = J_2(n) \mathbf{1}_{n=m}/3$.*

*Proof.* The nonzero Fourier coefficients are $\widehat{\psi}(r) = -1/(2\pi i r)$. The common frequencies of $f_h$ and $f_k$, followed by Parseval, give

$$\int_0^1 f_h f_k \, dx = \frac{\gcd(h,k)^2}{3hk}.$$

Indeed write $h = da$, $k = db$, $(a,b) = 1$. Matching frequencies requires indices $br, ar$, and the sum is $\sum_{r\neq 0}(\pi^2 abr^2)^{-1} = 1/(3ab)$. Divisor inversion gives $\sum_{d|n} J_2(d) = n^2$. Insert this identity in the Gram sum and exchange finite sums to obtain (57). The weighted coordinates of $g_n$ are $q_d(g_n) = \sum_{d|h|n} \mu(n/h) = \mathbf{1}_{d=n}$. Inversion proves the expansion and the asserted pairing. □

The factorization is the classical divisor factorization of a gcd matrix [12]. In this field basis its positive weights have an arithmetic meaning. $J_2(n)$ counts pairs $(a,b)$ modulo $n$ with $\gcd(a,b,n) = 1$. Inclusion and exclusion gives the displayed divisor sum. For $n = 2$ the three primitive pairs are $(0,1)$, $(1,0)$ and $(1,1)$.

For our four clocks,

$$(q_1, q_2, q_3, q_4) = (13/12, -1/4, 1/3, 1/4),$$
$$(J_2(1), \ldots, J_2(4)) = (1, 3, 8, 12).$$

Substitution gives continuous energy one, including every original clock interaction. Applying the same coordinates to $b = (2,0,0,0)$ and $r = (-1,-1,1,1)$ gives the selection energies stated above.

17.3. **The sampled Gram in the same coordinates.** Orthogonality belongs to an observation. For $g_1 = f_1$ and $g_2 = 2f_2 - f_1$, the three-point readings are

$$g_1 = (0, -1/3, 1/3), \qquad g_2 = (0, 1, -1).$$

Their continuous Gram is $\operatorname{diag}(1/3, 1)$, but their sampled Gram is

$$\begin{pmatrix} 2/27 & -2/9 \\ -2/9 & 2/3 \end{pmatrix}.$$

On this grid $g_2 = -3g_1$. A nonzero continuous field can therefore be invisible to these three readings. The coefficient change remains invertible; the sampling map does not. Keep the sampled Gram when using primitive coordinates for a sampling defect.

**Problem 39** (A reset mask and its source)**.** For the four-clock source above, retain reset denominators one and three. Reconstruct the selected source, find its continuous energy, and compute its mixed contribution with the complementary source. Compare with deleting the other two source coefficients.

**Problem 40** (An invisible primitive combination)**.** Use $H = 3g_1+g_2$. Find its continuous and three-point energies and its defect. Explain why a diagonal continuous Gram cannot replace the sampled Gram in this calculation.

## 18. Prescribed fields and quantitative recovery

An identity specifies the quantity to be estimated. Replacing it by a bound for every possible field can impose a much stronger requirement. For the matrix $\operatorname{diag}(T, -T)$, the vector $(1, 1)$ has quadratic reading zero although the operator norm is $|T|$. Large directions need not be the direction selected by the problem.

The distinction has an arithmetic instance in the kernel just constructed. It also appears when an exact inverse formula is used to recover individual prime-counting errors.

The reset selector in Section 17 provides a finite example. Its selected source has energy $4/3$, although the whole prescribed field has energy one. A bound on that selected part cannot replace the calculation of its interaction with the complement. The arithmetic norm example below makes the same distinction between control of every direction and control of the specified source.

18.1. **An interior operator-norm obstruction.** Let $H = \lfloor\sqrt{p}\rfloor$ and choose real weights $0 \le w_a \le 1$ that equal one on $H \le a \le p-H$. These weights modify the kernel coefficients; they are distinct from the row masses of a register. Form the symmetric matrix

$$R_w = (w_a w_b R_q(a, b))_{1\le a,b<p}.$$

Its prescribed scalar reading is $\mathbf{1}^T R_w \mathbf{1} = \sum_{a,b<p} w_a w_b R_q(a, b)$. Let $\Pi$ be orthogonal projection perpendicular to the constant vector. Removing that direction still leaves a large norm.

**Proposition 18.1** (Interior norm bound)**.** *For every odd prime $p \geq 257$ and every weight table as above,*

$$\|\Pi R_w \Pi\|_{\mathrm{op}} \geq p/64. \tag{58}$$

*Proof.* Choose the largest odd integer $r \leq \sqrt{p}/2$ and put

$$a = r^2, \qquad b = (p + r)/2, \qquad t = 2(p - r).$$

Then $p^2 = a + tb$ and $(a, b) = (t, a) = 1$. Both indices lie in the plateau. To see the lower bound on $a$, write $y = \sqrt{p} \geq 16$. Then $r > y/2 - 2$ gives $a > p/4 - 2y + 4 \geq y$. The other bounds follow from $a \leq p/4$ and $p/2 \leq b \leq p/2 + y/4$.

Reduction of $p^2 = a + tb$ modulo each denominator gives

$$R_q(a, b) = s(1, b) + s(t, a).$$

Here $s(\bar{t}, a) = s(t, a)$ by permuting the defining sum. Cauchy–Schwarz in that sum gives $|s(t, a)| \leq s(1, a)$. Summing the squares of $j/k - 1/2$ yields

$$s(1, k) = \frac{k}{12} - \frac{1}{4} + \frac{1}{6k}.$$

Since $b - a \geq p/4$, it follows that

$$R_q(a, b) \geq \frac{b - a}{12} + \frac{1}{6}(1/b - 1/a) \geq \frac{p}{48} - \frac{1}{6} \geq \frac{p}{64}.$$

The diagonal entries of $R_w$ vanish and $w_a = w_b = 1$. For $v = (e_a - e_b)/\sqrt{2}$, $\Pi v = v$, $\|v\| = 1$, and $v^T \Pi R_w \Pi v = -R_q(a, b)$. The definition of the operator norm proves the claim. □

Thus a uniform $O_\varepsilon(p^{1/2+\varepsilon})$ norm bound, for every $\varepsilon > 0$, is unavailable even on the mean-zero subspace. This does not decide the prescribed scalar sum. An operator bound must pay for directions that the scalar may never use. Retaining the prescribed source vector leaves its particular cancellation available to the estimate.

18.2. **Exact recovery and stability.** The logarithmic transform has a different issue. It can reconstruct a quantity exactly while enlarging a termwise error allowance. Keep the carrier $q = p^2$ fixed.

Since the first nonzero shell is $P_q(4) = 1/9$, the prime-counting identity at $N = 4n < p$ gives

$$\Psi(n) = 9\mathcal{K}_q(4n) - 9\sum_{m=5}^{4n} P_q(m)\Psi(4n/m). \tag{59}$$

For integer $n \geq 2$, every argument on the right has integer part less than $n$. The initial prime count is $\Psi(x) = 0$ for $x < 2$. Hence the formula is a finite triangular recovery, despite $P_q(1) = 0$ preventing an ordinary Dirichlet inverse.

To examine errors, retain the complete linear term:

$$F_q(N) = \mathcal{K}_q(N) - N\sum_{m\leq N} \frac{P_q(m)}{m}, \qquad E(x) = \Psi(x) - x.$$

Subtracting that term from (52) gives

$$F_q(N) = \sum_{m\leq N} P_q(m)E(N/m), \tag{60}$$

$$E(n) = 9F_q(4n) - 9\sum_{m=5}^{4n} P_q(m)E(4n/m). \tag{61}$$

These formulas include arguments between one and two, where $E(x) = -x$.

Suppose an induction tries to propagate the allowance $|E(x)| \leq Cx^\alpha$ by taking absolute values in (61). The $m = 5$ term alone has coefficient

$$9|P_q(5)|(4/5)^\alpha = \frac{5}{4}(4/5)^\alpha = (5/4)^{1-\alpha}. \tag{62}$$

For $\alpha < 1$ this exceeds one. At square-root scale it is $\sqrt{5}/2$. Thus that absolute induction does not contract. The signed recurrence remains intact; the calculation identifies the information discarded by the attempted estimate.

Figure 29 shows how this one term enlarges a sublinear allowance. Any successful use of the recurrence must recover information discarded by the absolute estimate.

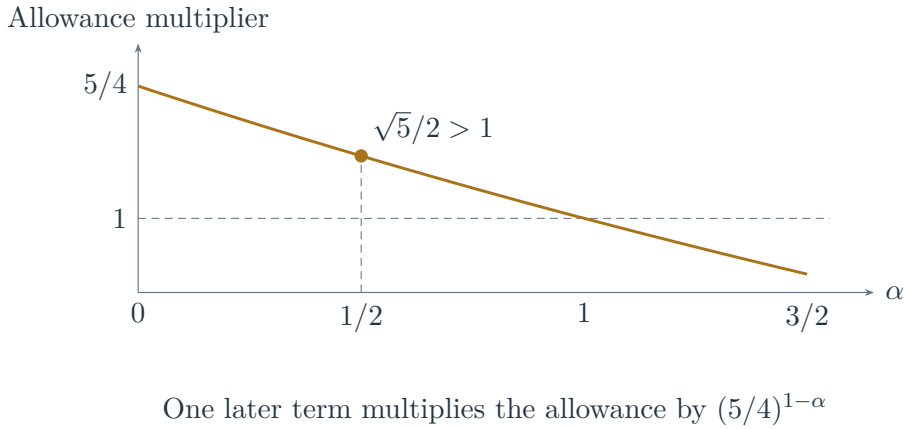



FIGURE 29. Exact recovery can enlarge an error allowance. The $m = 5$ term alone exceeds one at every exponent $\alpha < 1$. The graph concerns the termwise absolute induction. It does not determine the actual signed prime-counting error.

18.3. **Additional estimates for the arithmetic application.** The transform uses shorter prefixes at the same carrier. At $N = p-1$, the coefficient of $d = 1$ in (51) vanishes because $\log 1 = 0$. Every remaining prefix therefore has cutoff at most $(p-1)/2$. A bound on $\mathcal{A}_{p^2}(p-1)$ alone supplies neither those readings nor their logarithmically weighted sum. Replacing $p$ by a smaller prime also changes the kernel.

A quantitative prime-distribution application needs estimates for these fixed-carrier prefixes and control of the signed recovery in (61). Those estimates are open for the collision family considered here. The proved tools are the finite decomposition, the transform and the recovery identity. Their use has made the missing information explicit.

The same discipline applies to the register moment. Its finite pair formula retains all signs and observed intersections; an improved rate requires an estimate for their joined sum. The smoothing proposition gives the stated one-third normalized rate under its explicit analytic hypothesis. It supplies no stronger rate merely by changing the width.

**Problem 41** (A quantitative recovery test)**.** A triangular recurrence has first nonzero coefficient $P(a) = c \neq 0$ and a later coefficient $P(b) = d$, with $b > a$. What coefficient does the $b$ term contribute to a termwise allowance $Cx^{\alpha}$ after solving for the value at $n$ using the observation at $an$? Apply your answer to $a = 4$, $b = 5$, $c = 1/9$, $d = 5/36$.

**Problem 42** (A contracting recovery)**.** Let $E(x) = 0$ for $0 \leq x < 1$ and define $F(x) = E(x/2) - \frac{1}{4}E(x/4)$. Suppose $|F(x)| \leq C\sqrt{x}$ for every real $x \geq 2$. Recover $E(n)$, for real $n \geq 1$, as a finite sum of shorter observations of $F$, and derive an explicit square-root bound. Compute the contraction factor and compare it with the factor in Problem 41.

The distinction between recovery and stability also arises in *Finite Generation of Collision Capacity* [18]. There the complete capacity sequence determines a primitive boundary kernel within a specified symmetric regular class. A finite prefix does not determine that kernel, and extracting signed arithmetic readings has a separate sensitivity cost.

An invertible formula has now appeared in two roles. It can recover the required data, and it can amplify their errors. The next part makes that cost visible in small measured spaces, beginning with the least work needed to produce a retained reading.

Part 4

# Measured geometry and refinement

A reset table can recover a source, but the preceding calculation showed that an exact recovery can enlarge an error allowance. Keeping only part of a record introduces another question. Several sources may then give the same retained readings while carrying different energies. A record of differences may also lose a common level. To use such a record in an energy estimate, we must know which readings it determines and what can remain unseen.

Two source columns on three weighted addresses introduce a measured projection and its unseen directions. A four-address cycle then shows what a difference loses, what a return can reveal, and what refinement changes. Complete scale packets move cancellation onto the observation. Adjoint receivers put the same reading back on its source coefficients.

A record is sufficient only for the observations it can recover. The examples keep the tables small enough to test that sufficiency by hand. Missing means, unseen source directions and changed densities each have a calculable cost.

## 19. Measured projections and transported observations

Several sources can produce the same retained reading. Choosing the one with least energy requires the quadratic rule that computes that energy, including interactions between its components. This rule is the *work form*. This section extends the constrained minimum of Proposition 9.4 to sources whose components overlap. It also follows the observation through a change of coordinates. The underlying inner-product geometry, orthogonal projections and least-norm solutions are developed in [2, Chapters 6 and 7].

19.1. **Two sources on three addresses.** Take component columns $v_1 = (1, 0, 1)^T$, $v_2 = (0, 1, 1)^T$. Give the addresses masses $1, 2, 1$. For coefficients $c = (c_1, c_2)^T$, the field and its work are

$$Vc = (c_1, c_2, c_1 + c_2)^T, \qquad E[c] = c^* G c,$$

$$G = V^* \operatorname{diag}(1, 2, 1) V = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

At $c = (1, -1)^T$, the readings are $(1, -1, 0)^T$ and the energy is three. The shared third address contributes the off-diagonal entries. This

example uses a new finite observation, with total mass four. It is not the three-point clock grid of Section 17.

Let $c = Az$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z = (2, -1)^T.$$

The same field has Gram $G_z = A^*GA = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 7 \end{smallmatrix}\right)$ and energy $z^*G_z z = 3$. A detector reading $c_1$ becomes $z_1 + z_2$. Changing the coefficients while leaving either the Gram or the detector unchanged would describe another experiment.

**Proposition 19.1** (Transport of work and readings)**.** *For an invertible coordinate map $c = Az$, a Hermitian work matrix $G$ and a linear detector $L$, the transported data are*

$$G_z = A^*GA, \qquad L_z = LA.$$

*Equivalently, if $z = Wc$, then $G_z = W^{-*}GW^{-1}$ and $L_z = LW^{-1}$. In Euclidean operator norm,*

$$\|G_z\| \le \|W^{-1}\|^2 \|G\|, \qquad \|L_z\| \le \|L\| \|W^{-1}\|.$$

*Proof.* Substitute $c = Az$ in $c^*Gc$ and $Lc$. The same substitution in $c^*Gd$ preserves the complete mixed form. The norm inequalities follow from submultiplicativity and $\|W^{-*}\| = \|W^{-1}\|$. □

A general invertible change preserves the measured work with its transported matrix. It need not preserve the ordinary eigenvalues or trace of that matrix. The unitary statement in Proposition 9.1 has stronger hypotheses.

19.2. **Least-work reconstruction.** Fix the detector value $c_1 = 1$ in the two-source example. Then

$$E[(1, t)^T] = 2 + 2t + 3t^2 = \frac{5}{3} + 3(t + 1/3)^2.$$

The least-work source is $(1, -1/3)^T$, with readings $(1, -1/3, 2/3)^T$. The observed value does not require setting the unobserved coefficient to zero. The negative coefficient reduces work at the shared address.

**Proposition 19.2** (Least-work lift)**.** *Let $G$ be positive definite and let $U$ have independent rows. For a retained record $y$, define*

$$T = G^{-1}U^*(UG^{-1}U^*)^{-1}, \qquad \Pi = TU.$$

*Then $Ty$ is the unique minimum of $c^*Gc$ subject to $Uc = y$. Its work is $y^*(UG^{-1}U^*)^{-1}y$. Moreover*

$$U(I - \Pi) = 0, \quad \Pi^2 = \Pi, \quad \Pi^*G = G\Pi,$$
$$E[c] = E[\Pi c] + E[(I - \Pi)c].$$

*Proof.* Row independence makes $UG^{-1}U^*$ positive definite, since its quadratic form at nonzero $v$ is $(U^*v)^*G^{-1}(U^*v) > 0$. Multiplication gives $UT = I$, hence the first two projection identities. The expression $G\Pi = U^*(UG^{-1}U^*)^{-1}U$ is self-adjoint. If $Uc = y$, write $c = Ty + r$ with $Ur = 0$. Then $(Ty)^*Gr = y^*(UG^{-1}U^*)^{-1}Ur = 0$. Expansion gives the work formula and strict positivity gives uniqueness. Taking $y = Uc$ proves the final identity. □

For our detector $U = (1, 0)$, $T = (1, -1/3)^T$ and $\Pi = \left(\begin{smallmatrix} 1 & 0 \\ -1/3 & 0 \end{smallmatrix}\right)$. The remainder of $c = (1, -1)^T$ is $(0, -2/3)^T$. Its work is $4/3$, so the decomposition is $3 = 5/3 + 4/3$. Orthogonality here uses $G$.

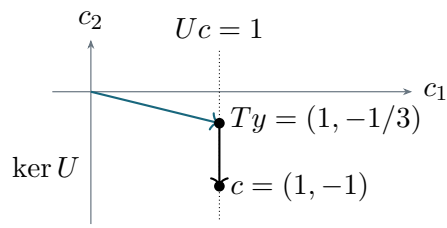



FIGURE 30. The retained reading fixes a vertical line of possible sources. The least-work lift is orthogonal to that line in the measured pairing $G$. Ordinary coordinate distance does not display this orthogonality.

The vertical direction leaves the detector unchanged. The minimum is perpendicular to it in the measured pairing, although these ordinary coordinate axes do not draw that pairing as Euclidean distance.

19.3. **Null directions and the observable quotient.** Duplicate the first component and use the three source columns $v_1, v_2, v_1$. Sources differing by a multiple of $(1, 0, -1)^T$ produce identical readings and identical work. The Gram is now semidefinite. Its inverse does not exist on the full coefficient space.

The *observable quotient* identifies sources differing by a vector in $\ker V$. Positive address weights give $\ker(V^*WV) = \ker V$, because zero work is equivalent to every weighted reading being zero. The work is positive definite on that quotient. A detector descends to it exactly when it annihilates $\ker V$. The first-address reading $c_1 + c_3$ does; the coefficient reading $c_1$ does not. The least-work formula applies after choosing coordinates on the quotient, provided the retained rows are independent there. It then determines a unique observable class, not a unique representative of the original redundant source.

19.4. **A moved boundary and its density.** Coordinate transport also acts on a continuous observation. Let $u \in [0, 1]$ and move a boundary at $u = 1/4$ to $v = 1/2$ by

$$v = \Theta(u) = \begin{cases} 2u, & 0 \le u \le 1/4, \\ (2u + 1)/3, & 1/4 \le u \le 1. \end{cases}$$

The inverse density is $du/dv = 1/2$ on the first half and $3/2$ on the second.

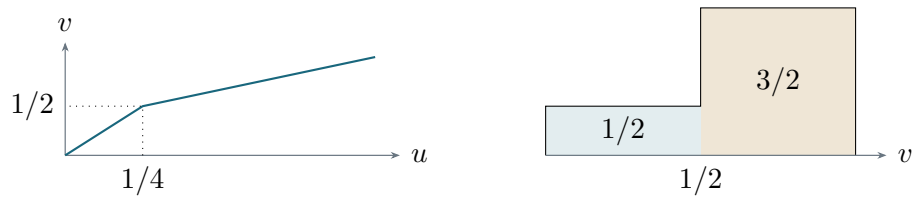



FIGURE 31. Moving the boundary changes the observation density. The left graph sends $1/4$ to $1/2$; the right graph carries masses $1/4$ and $3/4$ on the two new half-intervals. The density preserves the continuous energy.

The left graph moves the boundary to the midpoint. The right graph shows the transported density, whose two half-cell masses are $1/4$ and $3/4$.

A field equal to one before the boundary and two after it has work $1/4 + 4(3/4) = 13/4$. In the new coordinate,

$$\int_0^1 |f(\Theta^{-1}(v))|^2 (\Theta^{-1})'(v)\, dv = \frac{1}{2}\frac{1}{2} + 4\frac{3}{2}\frac{1}{2} = \frac{13}{4}.$$

Discarding the density gives $5/2$. The map fixes both endpoint samples, but fixing samples alone does not fix a continuous integral. An assigned value at the moved boundary is transported as a point value too. This small calculation is the local operation used by a common reset coordinate.

**Problem 43** (A different retained reading)**.** In the two-source example retain $c_1 + c_2 = 1$. Find the least-work source, its three readings and its work. Verify orthogonality to the detector's null direction. Then express the detector in the $z$ coordinates used above.

**Problem 44** (Transporting a cell)**.** Move a boundary at $0 < \theta < 1$ to the midpoint by a map affine on each side and fixing zero and one. Derive the inverse density. For readings $a, b$ on the two open pieces, verify the energy identity and calculate the error from using unit density after the move.

## 20. Cyclic differences and recovery

A difference table retains changes between readings and loses their common level. Taking a second difference records how those changes themselves vary. Recovering the original table requires the missing mean and an inverse whose cost depends on the cycle length.

This discrete curvature acts on readings. The regular curvature in Theorem 5.1 acts on a continuous profile between jumps. They have related uses, but different domains and different kernels. Finite Fourier analysis diagonalizes the cyclic shift and makes the inverse cost explicit; [26] supplies the general theory.

20.1. **A four-address cycle.** Index a vector by $j = 0, 1, 2, 3$ and define the forward shift $(Sx)_j = x_{j+1 \bmod 4}$. Put $D = I - S$. For

$x = (1, 2, 0, 1)^T$,

$$Dx = (-1, 2, -1, 0)^T, \qquad D^2x = (-3, 3, -1, 1)^T, \qquad \bar{x} = 1.$$

Adding any constant to $x$ leaves both difference records unchanged. The curvature alone cannot choose between $x$ and $x + 10\mathbf{1}$.

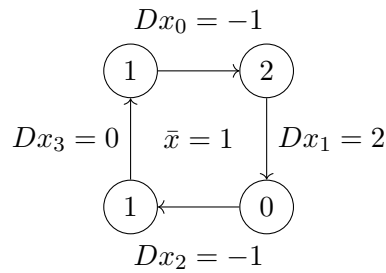



FIGURE 32. A four-address cycle and its forward differences $x_j - x_{j+1}$. The arrows follow the next address, including the closing step. Differences retain the changes but lose the common level; the mean must be recorded separately.

The arrows indicate the forward neighbor used in $x_j - x_{j+1}$. The cycle closes, so the differences sum to zero.

**Proposition 20.1** (Curvature and mean reconstruction)**.** *On a cycle of length $L \geq 2$, let*

$$\Pi = \frac{1}{L}\sum_{j=0}^{L-1} S^j, \qquad B = -\frac{I - \Pi}{L}\sum_{j=0}^{L-1} jS^j.$$

*Then $DB = BD = I - \Pi$, $B\Pi = 0$, and every table satisfies*

$$x = \Pi x + B^2D^2x. \tag{63}$$

*Its curvature and mean determine it uniquely. In counting norm, or in that norm divided by $\sqrt{L}$,*

$$\|B^2\|_{\mathrm{op}} = \frac{1}{4\sin^2(\pi/L)}. \tag{64}$$

*Proof.* Using $S^L = I$, multiplication gives $(I-S)\sum_{j=0}^{L-1} jS^j = L(\Pi - I)$. Multiplication by the negative factor in $B$ gives $DB = BD = I - \Pi$ and $B\Pi = 0$. Squaring proves (63). The cycle characters have shift eigenvalues $e^{2\pi ik/L}$. Only the constant character has eigenvalue

one. Thus $\ker D^2$ is the constant space, proving uniqueness. On a nonconstant character, $B^2$ has eigenvalue $(1-e^{2\pi ik/L})^{-2}$. Its greatest modulus is (64); the unitary Fourier basis proves the norm statement. □

The inverse second difference has cost of order $L^2$, since $\sin(\pi/L) \sim \pi/L$. This is an operator norm, not its square. An energy estimate obtained by squaring this norm can cost order $L^4$. Exact recovery and uniform stability are separate statements.

The eight-address factors in Figure 33 show where that inverse cost comes from. The constant mode disappears; the weakest surviving mode determines the largest amplification.

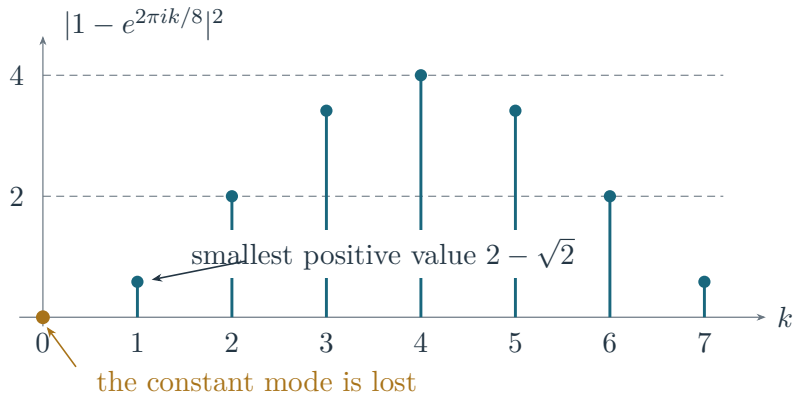



FIGURE 33. The amplitude factors of the second difference on an eight-address cycle. These are the moduli of its Fourier eigenvalues, not the eigenvalues themselves. The zero mode is the missing mean. Recovering the weakest nonconstant mode costs $1/(2-\sqrt{2})$ in norm. Longer cycles have a smaller first positive factor and a larger inverse cost.

For the four-address example the mean-zero table is $u = (0, 1, -1, 0)^T$. Direct substitution in $D^2u = (-3, 3, -1, 1)^T$ recovers it, and adding the mean gives $x$. If $k = D^2x$ has an error $e$ of mean zero and the supplied mean has error $\delta$, then the recovered error is

$$\delta\mathbf{1} + B^2e, \qquad \|\delta\mathbf{1} + B^2e\|_2^2 = L|\delta|^2 + \|B^2e\|_2^2.$$

The two parts are orthogonal. For a union of disjoint cycles, retain one mean on each cycle and use its own length in the inverse cost.

20.2. **Recovering an observation.** A detector $\ell$ reads

$$\ell x = \ell \Pi x + \ell B^2 k.$$

The row $\ell B^2$ is the detector transported to curvature data. Its norm bounds the effect of an error in $k$. For a quadratic observation $x^* H x$, substitute the whole expression $x = \Pi x + B^2 k$. Its mean-curvature mixed terms remain unless this particular $H$ makes them zero. Orthogonality in counting work does not imply orthogonality for every weighted observation.

**Problem 45** (A missing mean)**.** On the four-cycle, a table has curvature $(2, -2, 2, -2)^T$ and mean three. Recover the table. Change only the supplied mean to zero and compare the first-address reading and total counting energy.

## 21. Return histories and visibility

A single observation can miss a component that becomes visible after a return. Conversely, repeating an observation need not reveal every component. The return map and the detector together determine which source directions are seen and at what cost.

21.1. **A detector followed around a cycle.** Keep the four-cycle shift of Section 20 and use the two-component family with real coefficients $a, b$,

$$x = a\mathbf{1} + b\epsilon, \qquad \mathbf{1} = (1, 1, 1, 1)^T, \quad \epsilon = (1, -1, 1, -1)^T.$$

The return fixes $\mathbf{1}$ and reverses $\epsilon$. Let $\ell x = x_0$ be the detector. Its four successive readings are

$$(\ell x, \ell S x, \ell S^2 x, \ell S^3 x) = (a + b, a - b, a + b, a - b).$$

At $a = 1, b = -1$, the first reading is zero, but the full record is $(0, 2, 0, 2)$. Its mean square is two. In general that mean square is $a^2 + b^2$, whereas the counting work of the four-address field is $4(a^2 + b^2)$. The factor four belongs to the chosen normalization.

On this two-dimensional source, the polynomials

$$P_+ = (I + S)/2, \qquad P_- = (I - S)/2$$

recover $a\mathbf{1}$ and $b\epsilon$. They are projections here because $S^2 = I$ on this source. They are not the corresponding two projections on every four-address table, where $S^2$ need not be the identity. An operator relation must be attached to its domain.

**Proposition 21.1** (The complete history form)**.** *Let a finite-dimensional source have return $R$ with $R^L = I$. For a positive semidefinite observation matrix $H$, put*

$$G = \frac{1}{L}\sum_{j=0}^{L-1}(R^j)^* H R^j.$$

*Then $R^*GR = G$. Its nullspace is*

$$\ker G = \bigcap_{j=0}^{L-1} \ker(H^{1/2}R^j).$$

*This space is return-invariant. On the quotient by it, $G$ is positive definite and $R$ is an isometry.*

*Proof.* Conjugation by $R$ shifts the finite sum through one complete cycle, so its value is unchanged. Also $x^*Gx = L^{-1}\sum_j \|H^{1/2}R^j x\|_2^2$. The terms are nonnegative, so their sum vanishes exactly when all terms vanish. Advancing by $R$ permutes those conditions. The form therefore defines a positive inner product on the quotient and the same conjugation identity proves the isometry there. □

The individual event $H$ need not be preserved by the return. In the example, $H$ reads just the first address, and $SH \neq HS$. The complete history is what supplies an invariant work form.

21.2. **Return projections.** The finite Fourier sum gives a general version of the two-component projection. Work on the complexified history quotient, put $\zeta = e^{2\pi i/L}$, and define

$$P_k = \frac{1}{L}\sum_{j=0}^{L-1}\zeta^{-kj}R^j, \qquad 0 \le k < L.$$

Finite geometric sums give $P_kP_m = \mathbf{1}_{k=m}P_k$, $\sum_k P_k = I$ and $RP_k = \zeta^k P_k$. To check the first identity, collect the coefficient of each $R^s$ in the double sum. Its inner sum is zero unless $k = m$, when it equals $L$. Since $R$ is an isometry for $G$, eigenspaces for distinct unit eigenvalues are orthogonal in $G$. Thus these are measured orthogonal

projections. For a real source, conjugate eigenspaces combine into real invariant planes. This is finite Fourier analysis of the actual return, with the history observation retained.

21.3. **A nonzero reading and a lower work bound.** Recoverability can deteriorate while every finite observation remains invertible. On the same two-component source, use instead

$$\ell_\delta(a\mathbf{1} + b\epsilon) = a + \delta b, \qquad \delta > 0.$$

Two successive readings recover $a, b$ by their half-sum and their difference divided by $2\delta$. Their mean square is $a^2 + \delta^2 b^2$. Relative to source work $a^2+b^2$, the lower visibility constant is $\min(1, \delta^2)$. Every $\delta > 0$ gives recovery. A family with $\delta \to 0$ has no uniform positive lower constant.

As $\delta$ shrinks, the equal-work curves in Figure 34 widen along the poorly observed direction. The same observed work can hide a larger source.

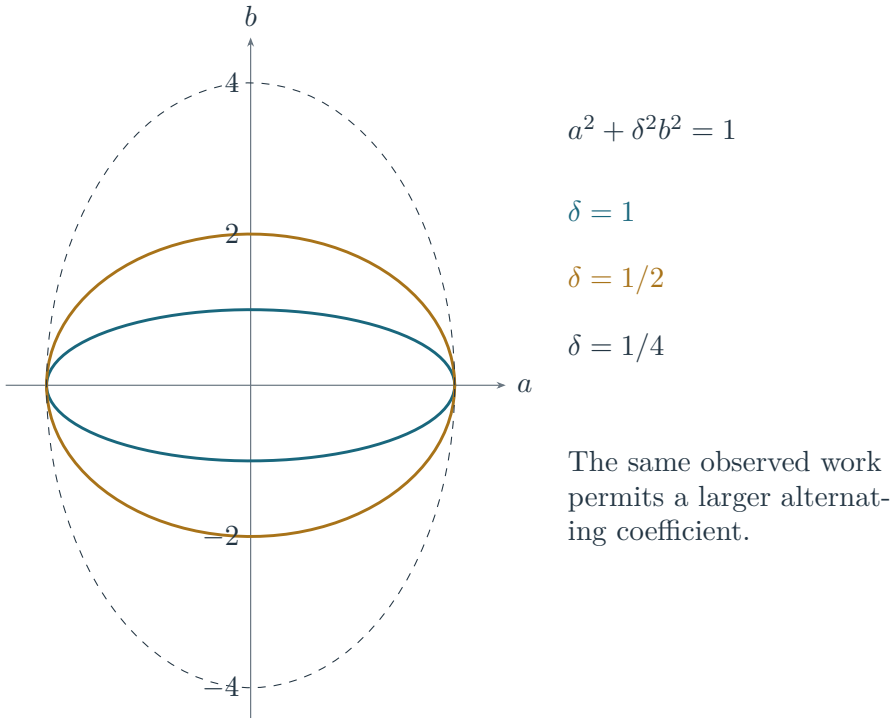



FIGURE 34. Sources with the same history work under three detector gains. The vertical axis is the alternating coefficient. As its gain falls, the observation allows a larger source in that direction. Recovery remains possible at each positive gain, while its uniform cost grows. The two coordinate axes use different display scales.

More generally, collect the rows of the full detector record in a matrix $O$. With a prescribed positive source work $G_0$, the normalized observation $L^{-1}\|Oc\|_2^2$ has lower constant

$$\lambda_{\min}\left(G_0^{-1/2} O^* O G_0^{-1/2}/L\right).$$

Substitution $z = G_0^{1/2} c$ proves this formula by the minimum Rayleigh quotient. A zero constant means an unseen direction. A positive constant at each scale still requires a lower bound uniform in that scale if uniform recovery is needed.

**Problem 46** (A detector losing one component)**.** Take $\delta = 1/2$ and $a = 1, b = -2$. Calculate the four detector readings, their mean-square work and the source work $a^2 + b^2$. Recover the source from the first two readings. Then take $\delta = 1/L$ and identify the cost of recovering the alternating coefficient as $L$ grows.

## 22. Refinement of a complete record

The full return history identifies the directions unseen at the present scale. Before discarding them, we must check whether they remain unseen under the next observation. Refinement is where that question arises.

Copying readings onto a finer grid preserves their energy when the sample masses are divided accordingly. It does not necessarily preserve what one step of motion means. Comparing the two orders, move then refine or refine then move, exposes the additional readings at the cuts.

22.1. **Two addresses refined to four.** Use the two cyclic advances

$$S_2(a, b) = (b, a), \qquad S_4(y_0, y_1, y_2, y_3) = (y_1, y_2, y_3, y_0).$$

Define

$$R(a, b) = (a, a, b, b), \qquad C(a, b) = (a, b, a, b).$$

The first map subdivides each old address into two consecutive copies. The second repeats the whole cycle. Both preserve mean-square work when masses $1/2$ become $1/4$. Their motion laws differ,

$$S_4^2 R = RS_2, \qquad S_4 C = CS_2.$$

For $R$, two fine steps correspond to one coarse step. For $C$, one fine step does. The maps encode different placements of the same data.

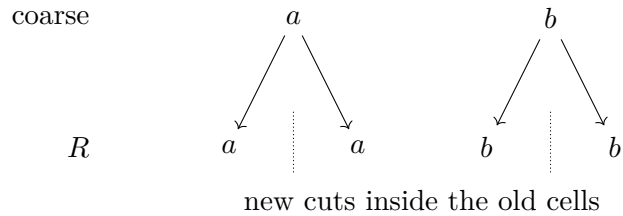



FIGURE 35. Literal subdivision copies each old reading into two adjacent addresses. Dividing its mass between the copies preserves mean-square work. One coarse step becomes two fine steps, so equal step counts need not preserve the reading.

For equal step counts the failure to commute is

$$E = S_4R - RS_2, \qquad E(a,b) = (a-b, 0, b-a, 0).$$

At $(a,b) = (1,-1)$ it is $(2,0,-2,0)$. The discrepancy is supported at two fine addresses and vanishes for a constant source. The support locates the change; its signed amplitudes determine work.

**Proposition 22.1** (Transported refinement error)**.** *Let $R : V \to W$ refine a source, and let $S$ on $V$ and $T$ on $W$ be its two advance maps. Put $E = TR - RS$. For every $m \geq 1$,*

$$T^mR - RS^m = \sum_{j=0}^{m-1} T^{m-1-j} E S^j. \tag{65}$$

*For a further refinement $R' : W \to Z$ with advance $T'$ and $E' = T'R' - R'T$, its composed error is*

$$T'R'R - R'RS = E'R + R'E.$$

*If both advances are isometries in specified norms, then $\|T^mR - RS^m\| \leq m\|E\|$.*

*Proof.* In the sum, replace $E$ by $TR - RS$. Adjacent terms cancel, leaving the two endpoints. For composition insert and subtract $R'TR$. Triangle inequality and the isometry bounds give the last assertion. These identities retain the complete maps before any observation is applied. □

In our example, two steps give

$$S_4^2R - RS_2^2 = R(S_2 - I),$$

which sends $(a, b)$ to $(b - a, b - a, a - b, a - b)$. It is the sum of two transported copies of $E$, not two errors at the same addresses. Squaring such a sum retains their mutual interaction. The norm allowance $m\|E\|$ is an accumulated upper bound; cancellation between its terms requires its own calculation.

22.2. **A direction unseen before refinement.** Suppose the coarse detector records only $O(a, b) = a + b$. The source $(1, -1)$ has reading zero. After refinement, the detector $O'y = y_0$ reads $O'R(1, -1) = 1$. Identifying the source with zero before refinement would lose this reading.

For general linear maps, the refined detector $O'R$ is determined by the old record $Oc$ exactly when

$$\ker O \subseteq \ker(O'R).$$

Necessity follows by testing an old null source. Conversely this inclusion makes $\widetilde{O}(Oc) = O'Rc$ well-defined on $\operatorname{im} O$, because any two representatives differ by a vector in $\ker O$. It defines the required linear detector there. This criterion explains when the present observable quotient can be carried forward. Until it is checked, transport the full source.

**Problem 47** (A compatible refined observation)**.** Keep $R(a, b) = (a, a, b, b)$ and the old record $a+b$. Compare refined detectors $O_1'y = (y_0 + y_2)/2$ and $O_2'y = y_0 - y_1$. Determine whether each factors through the old record and find the factor. Compute $E(a, b)$ and its mean-square work at $(a, b) = (2, 1)$.

## 23. Scale differences and complete packets

The refinement calculation retained a sum of transported differences. Its upper bound added their sizes. To use their signs, we need relations between the contributions before taking absolute values. A complete family of multiplicative scales supplies one such relation.

Multiplying an address by a prime becomes a translation after taking its logarithm. The signed family is then a finite difference of one profile. We can calculate which observations it cancels, how

a changing observation costs derivatives, and what survives when a cutoff removes one of the members.

23.1. **Two scale directions.** For $a > 0$ define $(T_a f)(z) = f(z + a)$ and $\Delta_a = I - T_a$. This notation denotes a forward translation difference on the real line, distinct from the backward integer difference in Section 10. For $a = \log 2$ and $b = \log 3$,

$$\Delta_a \Delta_b f(z) = f(z) - f(z + a) - f(z + b) + f(z + a + b).$$

If $f(z) = H(e^z)$, these are the readings of $H$ at scales $s, 2s, 3s, 6s$. The coefficients $1, -1, -1, 1$ are the divisor signs of the squarefree product six. Their sum is zero. Their first translation moment is also zero,

$$0 - a - b + (a + b) = 0.$$

Thus the difference annihilates constant and affine $f$. Applied to $f(z) = z^2$, it is $2ab$. Smoothness controls what survives after the first two terms cancel.

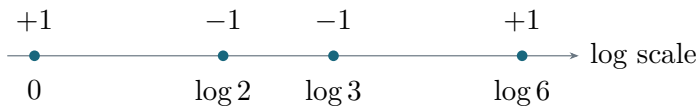



FIGURE 36. The complete scale packet at $s, 2s, 3s, 6s$, drawn in logarithmic coordinates relative to $s$. Its signs sum to zero and their first translation moment also vanishes. Removing a member changes the packet.

The spacings are logarithmic. The four signs belong to the complete packet; removing a member changes both moment sums.

**Proposition 23.1** (Translation differences)**.** *Let $a_1, \dots, a_r > 0$ and $f \in C^r(\mathbb{R})$. Then*

$$\prod_{i=1}^{r} \Delta_{a_i} f(z) = (-1)^r \int_0^{a_1} \cdots \int_0^{a_r} f^{(r)}(z + t_1 + \cdots + t_r)\, dt_r \cdots dt_1.$$

*Consequently its absolute value is at most $a_1 \cdots a_r \|f^{(r)}\|_\infty$ when that norm is finite. The difference annihilates polynomials of degree less than $r$.*

*Proof.* The fundamental theorem of calculus gives

$$f(z) - f(z+a) = -\int_0^a f'(z+t)\,dt.$$

Iterate this identity. The integrands are continuous on a compact box, so the integrations commute. Its volume is $a_1 \cdots a_r$. Taking absolute values gives the bound, and the derivative of a polynomial of lower degree is zero. □

The same operation can be transferred to the observation. For an integrable profile $f$ and a bounded test function $\phi$,

$$\int_{\mathbb{R}} \Delta_a f(z)\phi(z)\,dz = \int_{\mathbb{R}} f(z)\big(\phi(z) - \phi(z-a)\big)\,dz.$$

Change variables in the translated term to prove the identity. Repeated use moves all differences to $\phi$. If $f$ is integrable and $\phi \in C^r$ has bounded derivatives through order $r$, the resulting reading has the bound

$$\left| \int_{\mathbb{R}} \left( \prod_i \Delta_{a_i} f \right) \phi \right| \le \|f\|_1 a_1 \cdots a_r \|\phi^{(r)}\|_\infty.$$

The profile's signed difference is now charged to the actual observation's variation.

23.2. **A compact kernel and its moments.** An arithmetic profile need not be integrable before taking a difference. A step illustrates how the correct moment count changes. Let $h(z) = \mathbf{1}_{z<0}$ and take two lengths $a, b > 0$. Its complete packet is the compact kernel

$$K_{a,b}(z) = \Delta_a \Delta_b h(z) = \mathbf{1}_{[-b,0)}(z) - \mathbf{1}_{[-a-b,-a)}(z).$$

Endpoint values do not affect the integrals below. Direct integration gives

$$\int K_{a,b} = 0, \qquad \int z K_{a,b}(z)\,dz = ab.$$

There is one vanishing moment here. Two translation differences annihilate affine *input profiles*, while the resulting compact *step packet* has only its zeroth moment zero. These statements concern different objects.

For $\phi \in C^1$ with bounded derivative, substitution gives

$$\int K_{a,b}(z)\phi(z)\,dz = \int_{-b}^{0} \big(\phi(z) - \phi(z-a)\big)\,dz,$$

$$\left|\int K_{a,b}\phi\right| \le ab\|\phi'\|_\infty.$$

Figure 37 draws the case $a = b = 1$. Equal signed areas cancel the mass, while their separation leaves a nonzero first moment.

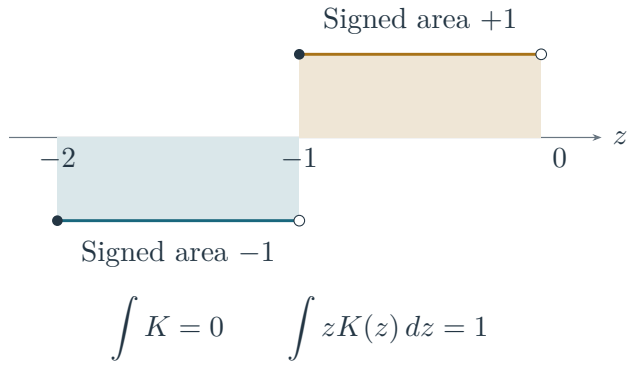



FIGURE 37. The step packet for $a = b = 1$. Equal areas cancel its mass, but their different locations leave a first moment of one. The compact kernel has one vanishing moment. This is distinct from the two translation differences annihilating affine input profiles.

One further translation difference produces $K_{a,b,c} = \Delta_c K_{a,b}$. Its constant and linear moments vanish, while

$$\int z^2 K_{a,b,c}(z)\,dz = 2abc, \qquad \left|\int K_{a,b,c}\phi\right| \le abc\|\phi''\|_\infty.$$

To verify the moment, subtract $(z-c)^2$ from $z^2$ under the $K_{a,b}$ integral and use its two known moments. For the bound, move the two differences onto $\phi$ and integrate over $[-b, 0)$. No arithmetic distribution hypothesis is needed for these finite translation identities.

The kernel can also act on a finite signed measure $\nu$. For bounded $\phi$ with the required bounded derivatives, Fubini gives

$$\left|\int (K_{a,b} * \nu)(z)\phi(z)\,dz\right| \le ab\|\nu\|_{\mathrm{TV}}\|\phi'\|_\infty.$$

The compact support of $K$ and finiteness of $\nu$ justify the interchange. With $K_{a,b,c}$ the derivative order is two and the factor is $abc$. This is how a packet identity becomes a bound for an observation. The total variation of its source measure remains part of the cost.

23.3. **Clipped packets.** A band mask $\chi(n)$ selects which members are actually observed. For a pair of contributions $A_m, A_{qm}$,

$$\chi(m)A_m + \chi(qm)A_{qm} = \chi(m)(A_m + A_{qm}) + [\chi(qm) - \chi(m)]A_{qm}.$$

The second term is the cost of an incomplete pair. For a four-member packet, index its contributions by $d \mid 6$. The complete identity is

$$\sum_{d|6} \chi(dm)A_{dm} = \chi(m) \sum_{d|6} A_{dm} + \sum_{d|6} [\chi(dm) - \chi(m)]A_{dm}.$$

At $m = 1$, the band $1 < n \leq 6$ removes the member at one. The signed coefficient sum of the retained members at $2, 3, 6$ is $-1$, whereas the full packet's sum is zero. A constant reading therefore survives this clipping. The incomplete packets at both ends of a longer band must be evaluated or bounded in their own observation. Smoothing a cutoff charges derivatives of the smoothing function; it does not remove that cost.

**Problem 48** (A complete packet and a missing member)**.** For $a = \log 2$, $b = \log 3$, compute the packet reading of $f(z) = 1$, $z$ and $z^2$. Repeat at $z = 0$ after removing its $f(0)$ term. For $a = b = 1$, write the two constant pieces of $K_{a,b}$ and verify its zeroth and first moments directly.

## 24. Dual readings and signed correlations

A linear reading can be calculated before the whole field is built. Move the detector back through the field map and it becomes a reading of the source coefficients themselves. The packet calculation used this operation to put differences on the observation. Here it exposes the signs in a source reading. The detector's size bounds every source direction; its response to the prescribed source can be much smaller. This is the usual adjoint construction in an inner-product space [2, Chapter 7], with the observation weights retained.

24.1. **The receiver on source coordinates.** Return to the columns $v_1, v_2$ and masses $1, 2, 1$ from Section 19. Let $d = (1, 1, 1)^T$ be a detector field. Its weighted linear reading is

$$\langle d, Vc\rangle_W = d^*WVc = (V^*Wd)^*c = (2, 3)c.$$

Here the bracket places the conjugated detector first. The *adjoint receiver* is $r = V^*Wd = (2, 3)^T$. For $c = (1, -1)^T$ the reading is $-1$. Directly, the observed field is $(1, -1, 0)^T$ and its weighted pairing with $d$ is $1 - 2 = -1$. Both orders use the same masses.

More generally, for any finite matrix $A$ and detector $d$, $d^*Ac = (A^*d)^*c$. This follows by expanding the finite sum and exchanging its two indices. When a weight matrix is present it belongs inside the adjoint, as it did in $V^*Wd$.

The receiver column $r$ represents the covector $r^*$. Its energy-dual source uses the positive Gram $G$,

$$u = G^{-1}r, \qquad r^*c = u^*Gc, \qquad |r^*c|^2 \le (r^*G^{-1}r)(c^*Gc).$$

Cauchy–Schwarz in the $G$ inner product proves the bound. For our receiver, $u = (3/5, 4/5)^T$ and $r^*G^{-1}r = 18/5$. The bound gives $|r^*c|^2 \le 54/5$ at the specified source, while its actual square is one. It controls every direction and does not use the source's particular cancellation.

24.2. **The square of a signed reading.** For a real receiver $r$ and a real source $s$,

$$(s^Tr)^2 = \sum_j s_j^2 r_j^2 + 2\sum_{j<k} s_j s_k r_j r_k. \tag{66}$$

If $s_j \in \{-1, 0, 1\}$ and $r_j = 0$ wherever $s_j = 0$, the diagonal is $\|r\|_2^2$. The complete signed square still contains the second sum. Here *covariance* names that deterministic mixed sum; no probability distribution or independence is assumed.

For $r = (1, 2, 1)^T$ and $s = (1, -1, 1)^T$, the reading is zero. The diagonal is six and the mixed term is $-6$. With the same receiver and source $(1, 1, 1)^T$, the square is sixteen and the mixed term is ten. Receiver norm alone cannot distinguish them.

Figure 38 compares the two sources before their contributions are squared. The same diagonal can accompany very different mixed work.

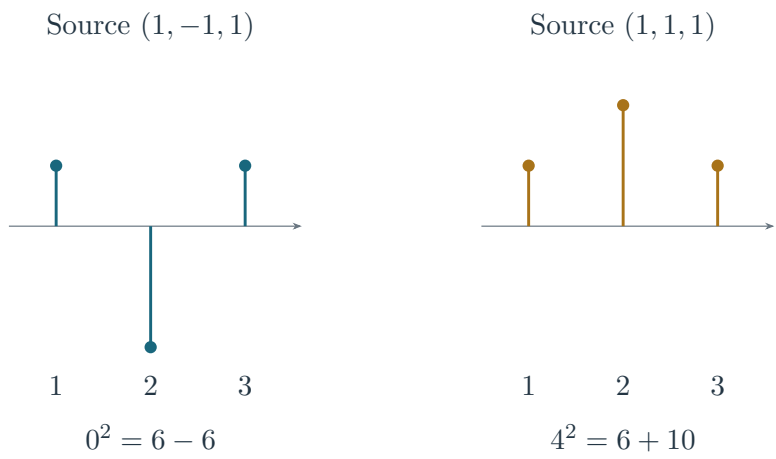



FIGURE 38. The same receiver has diagonal contribution six under both sign patterns. Stems show its signed contributions before they are added. Their mixed terms determine whether the squared reading is zero or sixteen. A receiver norm alone cannot distinguish the two sources.

At length $n$, the receiver $r_j = 1/n$ has squared norm $1/n$. All positive signs give reading one. For even $n$, alternating signs give reading zero. A small diagonal does not force the signed square to have the same order. The actual relation between signs and weights is the remaining information.

24.3. **Transporting an approximate receiver.** Suppose a finite calculation replaces $r$ by $\widetilde{r} = r + e$, with $|s_j| \le 1$. Then

$$|s^T\widetilde{r} - s^T r| \le \|e\|_1,$$
$$|(s^T\widetilde{r})^2 - (s^T r)^2| \le (2|s^T r| + \|e\|_1)\|e\|_1,$$
$$\left|\|\widetilde{r}\|_2^2 - \|r\|_2^2\right| \le (2\|r\|_2 + \|e\|_2)\|e\|_2.$$

The first inequality is triangle inequality. The second expands the perturbed scalar square, and the third expands the vector square and applies Cauchy–Schwarz. If both receivers vanish at the zero signs, subtracting their diagonal terms in (66) bounds the change in the mixed term by the sum of the last two allowances. A linear error allowance cannot be reused unchanged for its square.

24.4. **Energy in a short interval.** Recovering a slope from local energy also requires the observed interval. A shrinking interval can have decreasing work while its field becomes steeper. The mean height and slope enter separately.

For the real affine field $f(x) = a + bx$ on $0 < x < h$,

$$\int_0^h (a + bx)^2 \, dx = h(a + bh/2)^2 + \frac{b^2 h^3}{12}. \tag{67}$$

Write $a + bx = (a + bh/2) + b(x - h/2)$. The second term has mean zero; integrating its square gives $b^2h^3/12$, proving the identity. The first term records the mean height and the second the slope. Both are nonnegative. At $a = -bh/2$ the mean height is zero but the slope energy remains.

The formula gives $|b| \le \sqrt{12}\, h^{-3/2} \|f\|_{L^2(0,h)}$. Its cost grows as the interval shrinks. At $h = 1/N$, $a = -1/2$ and $b = N$, the work is $1/(12N)$ while the slope is $N$. An assigned reading at zero remains separate from the trace $a$; it contributes only to an observation with mass there.

Figure 39 compares two of these energy densities.

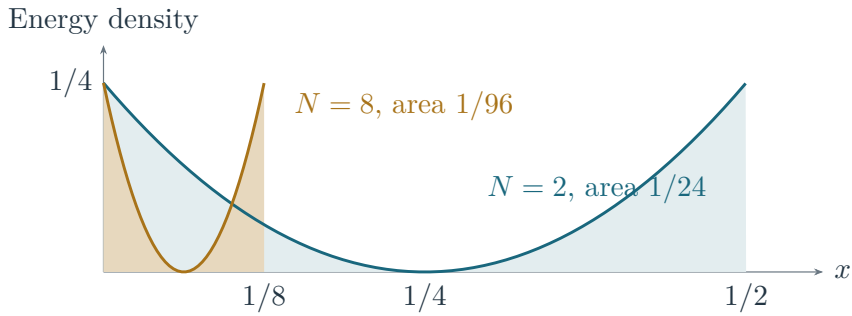



FIGURE 39. The density $(Nx-1/2)^2$ on $0 < x < 1/N$. Each curve has endpoint height $1/4$, but the interval shrinks as the field's slope grows. The complete area is $1/(12N)$. Small local energy alone therefore does not bound the slope independently of the observation length.

**Problem 49** (Signed receiver and perturbation)**.** Use $s = (1,-1,1)^T$, $r = (1,2,1)^T$ and $e = (0,\delta,0)^T$. Compute the new signed reading, its square, the new diagonal and the new mixed term. Verify all three error allowances above. For the affine field, find the smallest local energy at fixed $b, h$ when $a$ can vary.

Part 5

# Arithmetic energy at growing scales

The preceding parts give operations for an exact finite account. A growing arithmetic field also requires estimates whose constants are controlled as the scale changes. This final part develops two applications around that requirement.

In Sections 25 and 26, a carry table supplies a periodic clock field. Holding that field fixed while shifting its grid changes the simultaneous boundary readings. The boundary calculation identifies the contribution, and a separate signed estimate establishes its limit along the growing family.

Sections 27–29 follow a divisor field whose source and activation cutoff both change. The question is an upper bound for its complete energy increment. Screening pays for short completed cycles. Localization retains the whole tail within an allowance. The parent shell, combined resets and invariant readings then describe the work that is still present in that same increment. The application ends at the uniform estimate these operations have not yet supplied.

## 25. Carry tables and clock families

The carry table of Section 1 can now be read as a clock field. Recall that in base three, the equal-digit words $00, 11, 22$ have numerical values $0, 4, 8$. They pick out three positions among the nine possible two-digit words. More generally, a repeated digit $dd$ in base $p$ has value $(p+1)d$. The carry count defined below tests these selected positions at each integer address $a$. A floor difference counts the integers crossed between $na/q$ and $(n+1)a/q$. Centering removes the term $a/p$ so that the remaining fluctuation can be compared with a clock field.

Let $p$ be an odd prime, $N = p-1$ and $q = p^2$. The equal-digit base-$p$ words occupy the positions $G_p = \{(p+1)d : 0 \le d < p\}$ on a circle of $q$ positions. Their carry count and centered reading are

$$C_p(a) = \sum_{n \in G_p} \left( \left\lfloor \frac{(n+1)a}{q} \right\rfloor - \left\lfloor \frac{na}{q} \right\rfloor \right), \qquad F_p(a) = C_p(a) - \frac{a}{p}.$$

Write $E_p = \sum_{a=0}^{q-1} F_p(a)^2$ and $B_N(x) = 2\sum_{j=1}^{N} \psi(jx)$. For $p = 3$ and $a = 1$, the three floor differences are $0, 0, 1$, so $C_3(1) = 1$ and $F_3(1) = 2/3$. The next lemma identifies every centered table reading with a reading of $B_2$, after permuting the nine addresses. It therefore

transfers the entire squared table, not just its average, to the clock calculation.

**Lemma 25.1** (Carry representation)**.** *If* $[z]_q$ *is the least residue modulo* $q$*, then*

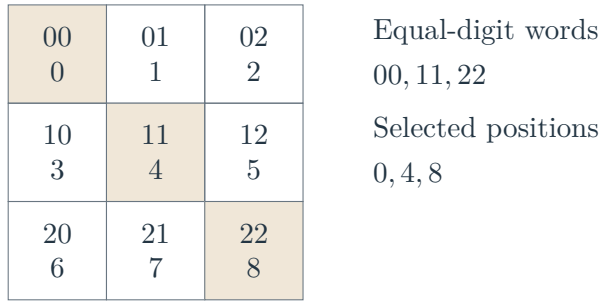
$$F_p([(1-p)A]_q) = B_N(A/q), \qquad E_p = \sum_{A=0}^{q-1} B_N(A/q)^2.$$

*Proof.* Expanding each floor as its argument minus its fractional part cancels the allowance $a/p$. At $a = [(1-p)A]_q$, the congruences $(1-p)(p+1)d \equiv d \pmod q$ and $(1-p)((p+1)d+1) \equiv d+1-p \pmod q$ give

$$F_p(a) = \sum_{j=1}^{p-1} \big(\{jA/q\} - \{-jA/q\}\big) = 2\sum_{j=1}^{p-1} \psi(jA/q).$$

The equality includes integer arguments. Multiplication by $1-p$ is a permutation modulo $q$, proving the energy identity. □

Figure 40 makes the base-three selection and address permutation explicit. The relabeling carries every reading into the clock representation.




| $A$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | 0 | 7 | 5 | 3 | 1 | 8 | 6 | 4 | 2 |

$a = [-2A]_9$ relabels every address once

FIGURE 40. The base-three carry table begins with three selected two-digit words. Each square shows a word above its numerical value. The lower rows give the permutation that identifies the centered carry readings with the two-clock field. The complete energy is preserved under this relabeling.

At a reduced boundary $a/k$, the clocks whose frequencies are multiples of $k$ reset together. There are $\lfloor N/k \rfloor$ of them. Grouping the other frequencies into blocks of length $k$ cancels each complete block; only the final incomplete block contributes to the inherited reading. The next notation records the number of complete blocks, the remaining length and that remaining reading. Each reduced fraction is counted once by requiring $a$ and $k$ to be coprime.

For $2 \le k \le N$, set

$$M_k = \lfloor N/k \rfloor, \quad s_k = N - kM_k,$$

$$T_{k,s}(a) = 2\sum_{j=1}^{s} \psi(ja/k), \quad H_N = \sum_{j=1}^{N} \frac{1}{j}.$$

Let $\mathcal{U}_k = \{1 \le a < k : (a,k) = 1\}$, and put

$$\mathcal{A}_p = \sum_{k=2}^{N} M_k \sum_{a \in \mathcal{U}_k} T_{k,s_k}(a)\psi(p^2 a/k).$$

**Proposition 25.2** (Carry boundary identity)**.** *The carry sampling defect satisfies*

$$D_{p^2}[B_{p-1}] = \frac{N^2}{p^2} + \frac{4\mathcal{A}_p}{p^2} - \frac{NH_N}{3p^3}. \tag{68}$$

*Proof.* The common reset at zero has $C_0 = N$ and $U_0 = 0$. At a reduced interior boundary $a/k$, the resetting clocks are the multiples of $k$, so $C_{a/k} = M_k$. The nonreset readings in each complete block of $k$ clocks sum to zero, leaving $U_{a/k} = T_{k,s_k}(a)$. No interior boundary lies on the $p^2$ grid, because $k < p$ and $(k, p^2) = 1$. The first two terms of (7) are therefore $N^2/p^2$ and $4\mathcal{A}_p/p^2$.

The slope is $V = N(N+1) = pN$. To evaluate the last term, use

$$\begin{aligned}\sum_{b\in\mathcal{B}} C_b \widetilde{B}_2(p^2 b) &= \frac{N}{6} + \sum_{k=2}^{N} M_k \sum_{a\in\mathcal{U}_k} \widetilde{B}_2(p^2 a/k) \\ &= \sum_{j=1}^{N}\sum_{a=0}^{j-1} \widetilde{B}_2(p^2 a/j) = \sum_{j=1}^{N} \frac{1}{6j} = \frac{H_N}{6}.\end{aligned}$$

The middle identity counts each reset once for each clock that resets there. Multiplication by $p^2$ permutes residues modulo every $j < p$. Finally,

$$\sum_{a=0}^{j-1} \widetilde{B}_2(a/j) = \frac{1}{6j},$$

obtained by summing the quadratic polynomial using the sums of integers and their squares. Substitution in (7) gives the harmonic term. □

The continuous energy can also be read from pairwise clock collisions. The Fourier coefficients of $\psi$ are $-1/(2\pi i n)$ for $n \neq 0$. The frequencies common to $\psi(jx)$ and $\psi(kx)$ give, by Parseval,

$$\int_0^1 \psi(jx)\psi(kx)\,dx = \frac{(j,k)^2}{12jk}, \qquad \int_0^1 B_N^2\,dx = \frac{1}{3}\sum_{j,k\le N} \frac{(j,k)^2}{jk}.$$

Together with Lemma 25.1, this recovers both sides of the carry-energy comparison from the same finite family.

For $p = 3$, $\mathcal{A}_3 = 0$ and $H_2 = 3/2$, giving $D_9[B_2] = 4/9 - 1/27 = 11/27$. In general, the established unit-sampling theorem for this

carry family [21] supplies the separate estimate, for every $\varepsilon > 0$,

$$D_{p^2}[B_{p-1}] = 1 + O_\varepsilon(p^{-1/4+\varepsilon}).$$

Equation (68) identifies its residual as $4\mathcal{A}_p/p^2 - NH_N/(3p^3)$. The boundary decomposition does not replace the signed estimate proving that this residual tends to zero. The finite common-reset contribution is $(p-1)^2/p^2$.

25.1. **Restoring the common reset.** The prescribed field reads zero at the origin, while both limiting energies there are $N^2$. Restore that one energy reading in the sample average and write

$$E_p^{\text{corrected}}/p^2 = E_p/p^2 + N^2/p^2.$$

Point-value replacement and the quoted estimate give

$$\int_0^1 B_N^2\,dx - E_p^{\text{corrected}}/p^2 = O_\varepsilon(p^{-1/4+\varepsilon}).$$

The unit limit measures a persistent boundary contribution. Its rate measures the signed work left after that contribution is restored. These are two distinct parts of the same energy comparison.

The revised conductor proof in [21] bounds the source by

$$|\mathcal{A}_p| \ll_\varepsilon p^\varepsilon \left[p^{3/2}(\sqrt{VJ}+1) + p^2/V + p^2/J\right], \qquad 1 \le V, J \le p.$$

Here $V, J$ are frequency cutoffs, unrelated to the clock slope $V$ in Section 6. Choosing both near $p^{1/4}$ gives $\mathcal{A}_p = O_\varepsilon(p^{7/4+\varepsilon})$, which yields the rate after division by $p^2$. The product of the square of the first main allowance and the two tail allowances is $p^7$. Their maximum is therefore at least $p^{7/4}$ for every choice of cutoffs. Better cutoff choices alone cannot improve this particular bound. This optimization concerns the allowances, not a lower bound for the actual signed source.

At $p = 5$, the continuous energy is $23/9$ and $E_p/p^2 = 48/25$. The defect is $143/225$. Its common-reset part is $16/25$, leaving corrected difference $-1/225$. The reset term is already larger than the whole discrepancy at this finite base, because the residual is negative. The half-cell observation in Section 26 has its own signed source and its own analytic transfer.

**Problem 50** (A base-five carry table)**.** Take $p = 5$, so $q = 25$ and $N = 4$. For $A = 0, 1, 5, 12$, compute $a = [(1 - p)A]_{25}$ and evaluate $C_5(a)$ directly from the floor sum. Compare $F_5(a)$ with $B_4(A/25)$. Explain why this address change preserves the complete squared table sum, even though the two formulas need not agree at the same address.

## 26. Half-cell carry grids

At the origin all $p-1$ clocks reset together. A half-cell shift misses that point and samples the reset at one half, where only the even-frequency clocks reset. The number of resetting clocks is halved, so their squared amplitude is quartered. This identifies the candidate change in the limit. The rest of the section calculates the other boundary terms and controls their combined contribution.

Retain the field $B_N$, with $N = p-1$, but sample it at $(a+1/2)/p^2$. This changes the sampling rule while leaving the continuous energy fixed. It is a shifted observation of the clock field; no new interpretation of these points as integer carry inputs is required.

Put $M = N/2$ and define the shifted interior source

$$\mathcal{A}_p^{\mathrm{half}} = \sum_{k=2}^{N} M_k \sum_{a \in \mathcal{U}_k} T_{k,s_k}(a)\psi(p^2 a/k - 1/2).$$

The location $1/2$ is now sampled. Exactly the even-frequency clocks reset there. All the odd-frequency clocks have reading zero there as well, since $\psi(j/2) = 0$ for odd $j$. Consequently $C_{1/2} = M$, $U_{1/2} = 0$, and $B_N(1/2) = 0$.

Figure 41 compares these reset amplitudes at base five. It identifies the point contributions before the signed remainder is evaluated.

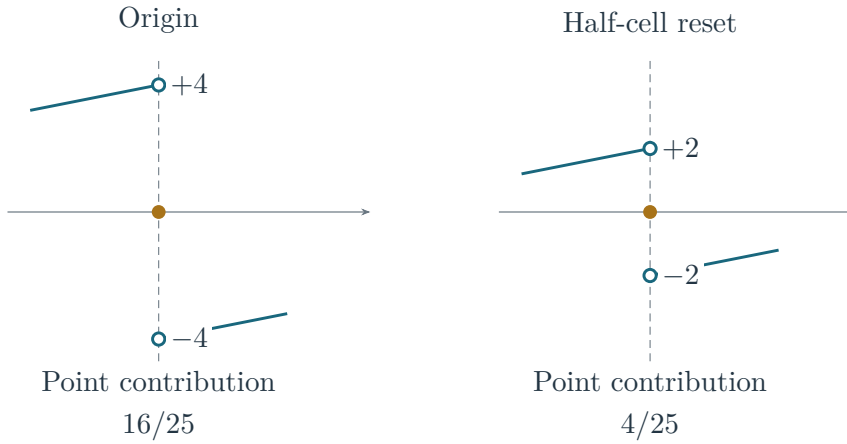



FIGURE 41. The same four-clock field near two sampled resets at $p = 5$. The unshifted grid reads the origin, where four clocks reset. The half-cell grid reads $1/2$, where two reset. Both assigned readings are zero. Halving the reset amplitude quarters its point contribution; the remaining boundary work still enters each full defect.

**Proposition 26.1** (Half-cell carry identity)**.** *For every odd prime $p$, with $N = p - 1$ and $M = N/2$,*

$$D_{p^2,1/2}[B_N] = \frac{M^2}{p^2} + \frac{4\mathcal{A}_p^{\mathrm{half}}}{p^2} - \frac{N(3H_M - 2H_N)}{12p^3}. \tag{69}$$

*The first term is the complete sampled point correction. The only sampled reset is at $1/2$.*

*Proof.* For a reduced boundary $a/k$, $k \le N$, membership in the grid requires $p^2 a/k - 1/2 \in \mathbb{Z}$. Since $(k, p^2) = 1$, this forces $k \mid 2$. Zero is not sampled, whereas $1/2$ is sampled because $p^2$ is odd. Thus the point correction is $M^2/p^2$. The value-jump term is $4\mathcal{A}_p^{\mathrm{half}}/p^2$ by the same boundary readings used in Proposition 25.2.

For the slope term, count boundaries by the clocks resetting there:

$$\sum_{b} C_b \widetilde{B}_2(p^2 b - 1/2) = \sum_{j=1}^{N} \sum_{a=0}^{j-1} \widetilde{B}_2(p^2 a/j - 1/2)$$
$$= \sum_{j=1}^{N} \frac{\widetilde{B}_2(j/2)}{j} = \frac{H_M}{8} - \frac{H_N}{12}.$$

The second equality uses the residue permutation modulo $j$ and

$$\sum_{a=0}^{j-1} \widetilde{B}_2(a/j + t) = \frac{1}{j} \widetilde{B}_2(jt).$$

This multiplication identity follows by summing the absolutely convergent Fourier series of $\widetilde{B}_2$: only frequencies divisible by $j$ remain. For the final equality, even $j$ have $\widetilde{B}_2(j/2) = 1/6$ and odd $j$ have $\widetilde{B}_2(j/2) = -1/12$. The even reciprocal sum is $H_M/2$. Substituting $V = pN$ into (7) proves the formula. □

For $p = 3$ the nine readings on the half-cell grid are

$$-\frac{5}{3},\ -1,\ -\frac{1}{3},\ \frac{1}{3},\ 0,\ -\frac{1}{3},\ \frac{1}{3},\ 1,\ \frac{5}{3}.$$

Their squares sum to eight. The continuous energy is one, so the defect is $1 - 8/9 = 1/9$, agreeing with the boundary calculation. In this example both the shifted interior source and the harmonic correction vanish.

## 26.1. **The half-cell limit.**

**Theorem 26.2** (Half-cell carry limit)**.** *As $p$ tends to infinity through odd primes,*

$$D_{p^2,1/2}[B_{p-1}] = \frac{1}{4} + O\big(p^{-1/458696} \log^4(2p)\big).$$

The grid shift changes which combined reset is sampled. Its surviving amplitude falls from $N$ to $N/2$. The two limits, one and one quarter, record the square of that change after the interior source has been estimated. The boundary calculation identifies the limit’s contribution; the parity transfer controls everything that remains.

The proof is completed at the end of this section, after the parity reduction and its analytic estimate. First compare the finite readings with the boundary contribution that predicts the limit.

26.2. **Finite readings of the shifted field.** The boundary term and the full defect approach their limits in different ways. The following values separate the sampled point contribution $M^2/p^2$ from the signed source $4\mathcal{A}_p^{\mathrm{half}}/p^2$. They are rounded displays of rational calculations made with *nfield* [19], software for exact arithmetic calculations.

Each row can be reconstructed by evaluating $B_N$ at the $p^2$ half-cell sample points, averaging their squares, and subtracting that average from the continuous Gram sum in Section 25. The boundary formula (69) gives an independent calculation of the same defect.

| $p$ | $M^2/p^2$ | $D_{p^2,1/2}[B_N]$ | $4\mathcal{A}_p^{\mathrm{half}}/p^2$ |
|---|---|---|---|
| 3 | 0.111111 | 0.111111 | 0.000000 |
| 5 | 0.160000 | 0.123556 | −0.035556 |
| 11 | 0.206612 | 0.101023 | −0.104967 |
| 31 | 0.234131 | 0.232033 | −0.001934 |
| 53 | 0.240655 | 0.259267 | 0.018684 |
| 61 | 0.241870 | 0.290206 | 0.048394 |
| 97 | 0.244872 | 0.206508 | −0.038337 |
| 101 | 0.245074 | 0.258963 | 0.013914 |

The sampled point correction increases toward one quarter. The full defect fluctuates on both sides of that value in the displayed range. For example, the boundary contributions at $p = 53$ and $p = 97$ are already close, but their signed sources have opposite signs. The table makes the role of the residual visible. Its uniform decay, proved in Theorem 26.2, establishes the limit.

Figure 42 places the two columns on common axes. The gap records the finite signed remainder.

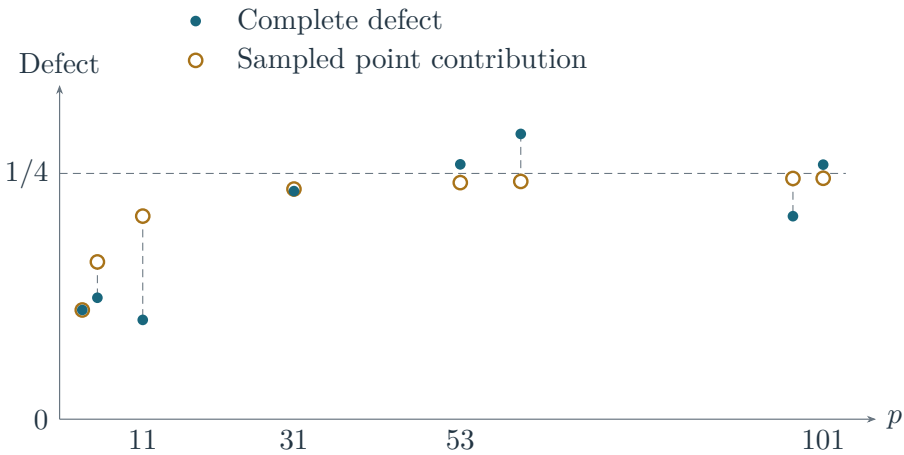



FIGURE 42. Finite half-cell defects and their sampled point contributions. Each vertical segment is the remaining signed boundary work, including the harmonic correction. The full defect can lie on either side of the point contribution. These exact finite calculations illustrate the terms; the uniform estimate in the theorem proves the limit.

**Problem 51** (A finite half-cell defect)**.** At $p = 5$, identify the sampled resets of $B_4$ on the half-cell 25-point grid and compute their combined point contribution. Evaluate $\mathcal{A}_5^{\mathrm{half}}$ by listing the nonzero incomplete blocks in its definition. Use the finite boundary identity to find $D_{25,1/2}[B_4]$. Compare the full answer with the point contribution.

26.3. **Residual criterion.** The point correction satisfies

$$\frac{M^2}{p^2} = \frac{(p-1)^2}{4p^2} \longrightarrow \frac{1}{4},$$

and the harmonic term is $O(p^{-2}\log(2p))$. As $p$ tends to infinity through odd primes,

$$D_{p^2,1/2}[B_{p-1}] \longrightarrow \frac{1}{4} \quad \Longleftrightarrow \quad \mathcal{A}_p^{\mathrm{half}} = o(p^2). \tag{70}$$

This equivalence distinguishes a limit for the sampled collision amplitude from a limit for the complete energy defect.

To compare the new source with the unshifted one, define for positive integers $N, Q$

$$\mathcal{A}_N(Q) = \sum_{k=2}^{N} \lfloor N/k \rfloor \sum_{a \in \mathcal{U}_k} T_{k,N \bmod k}(a)\psi(Qa/k).$$

There is no coprimality hypothesis in this definition. The original source is $\mathcal{A}_p = \mathcal{A}_{p-1}(p^2)$.

Subtracting a clock from its doubled version shifts its reset by half a cycle (Figure 43). This pointwise identity is the first step in the source reduction below.

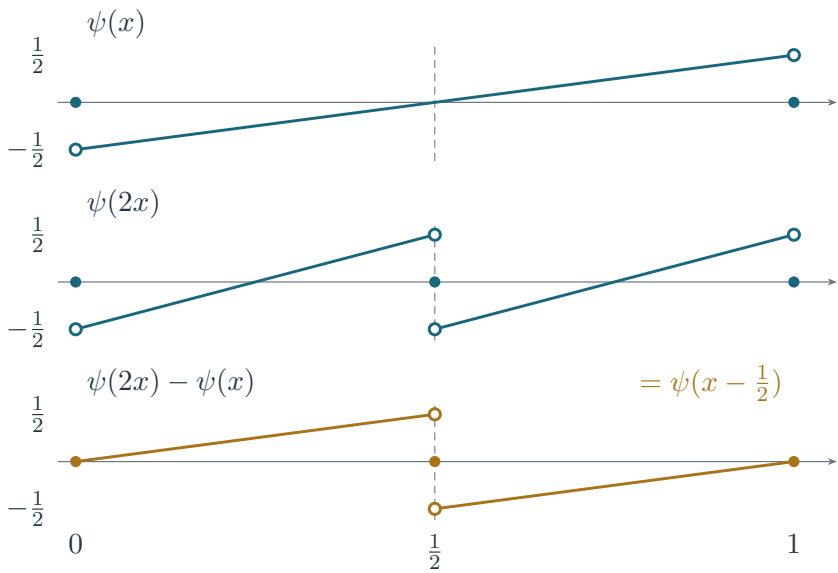



FIGURE 43. Subtracting the first clock from the doubled clock gives the half-shifted clock. Filled points are assigned readings and open points are one-sided limits. At integers and half-integers the difference is zero, so the identity includes the reset convention as well as the open segments. All rows share the same horizontal scale.

**Proposition 26.3** (Shift and parity reduction)**.** *For $N = p-1$ and $q = p^2$,*

$$\mathcal{A}_p^{\mathrm{half}} = \mathcal{A}_N(2q) - \mathcal{A}_N(q). \tag{71}$$

*For arbitrary positive integers $N, Q$, put*

$$R_k(u,v) = \sum_{a=0}^{k-1} \psi(ua/k)\psi(va/k).$$

*Then*

$$\mathcal{A}_N(Q) = 2 \sum_{l,j \le N} R_l(j,Q), \tag{72}$$

*and the even moduli obey*

$$R_{2k}(j,2q) = \begin{cases} 2R_k(j/2,q), & j \text{ even}, \\ R_k(j,q), & j \text{ odd}. \end{cases} \tag{73}$$

*Proof.* The integer-zero convention gives the pointwise identity

$$\psi(x-1/2) = \psi(2x) - \psi(x).$$

It follows by considering $0 < \{x\} < 1/2$ and $1/2 < \{x\} < 1$; at integers and half-integers both sides vanish. Substitution proves (71).

For (72), complete periods of $\psi(ja/k)$ sum to zero, so $T_{k,N \bmod k}(a)$ can be replaced by $2\sum_{j=1}^{N} \psi(ja/k)$ for $(a,k) = 1$. Expand $\lfloor N/k \rfloor$ as the number of multiples of $k$ through $N$. Reducing each fraction $a/l$, $0 \le a < l$, counts every reduced boundary with precisely that multiplicity. The boundary at zero contributes zero. This yields the full sum over $l, j$ without requiring an inverse of $Q$.

For (73), group the residues modulo $2k$ as $a = b$ and $a = b + k$. The second factor has the same value $\psi(qb/k)$ in both terms. If $j$ is even, the first factors are equal. If $j$ is odd, their sum is

$$\psi(jb/(2k)) + \psi(jb/(2k) + 1/2) = \psi(jb/k).$$

This is the same multiplication identity for $\psi$, valid at its midpoint readings. The two cases give the asserted reduction. □

The same shift also has a direct grid interpretation:

$$D_{q,1/2}[B_N] = 2D_{2q}[B_N] - D_q[B_N].$$

It follows from (12). This relation is exact for each fixed $N$. It does not supply an estimate for $D_{2q}$ along the growing family.

26.4. **Parity-restricted source.** The unshifted estimate controls $\mathcal{A}_N(q)$ for $N = p - 1$, $q = p^2$. Its count-source transformation uses the inverse of $q$ modulo every $l \le N$. The replacement $q \mapsto 2q$ is not permitted at even moduli. For instance, multiplication by 25 permutes the residues modulo four, while multiplication by 50 reaches only the even residues. An inverse exists for the first map and not for the second. Formula (73) removes that algebraic obstruction, and leaves three parity-restricted sums. Their cancellation can be transferred by retaining parity through the full reciprocity descent.

The residue maps in Figure 44 show the obstruction at modulus four. The reduction must retain multiplicities before applying an inverse.

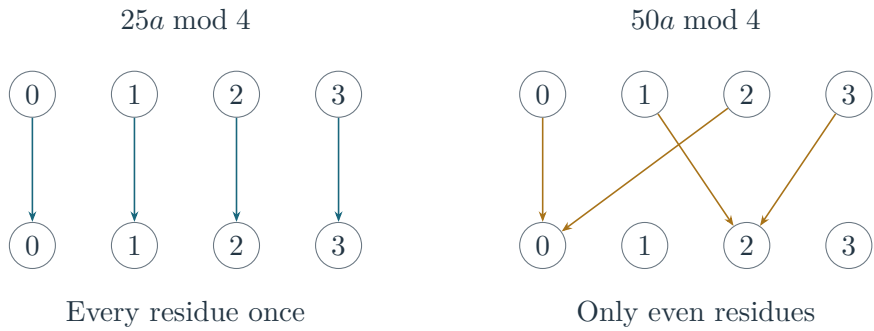



FIGURE 44. Doubling the carrier changes its arithmetic at even moduli. Modulo four, multiplication by 25 is invertible; multiplication by 50 has two readings at each even residue and none at the odd residues. The parity reduction retains these repetitions before any inverse is used.

Recall the Dedekind sum from Section 16. Since $\psi(0) = 0$, its equivalent form with the zero term included is $s(h,k) = \sum_{a=0}^{k-1} \psi(a/k)\psi(ha/k)$, with $s(h,1) = 0$. The argument $h$ need not be a unit. At odd $l$, $2q$ is invertible; at every $k \le M$, $q$ is invertible. Hence the remaining source is exactly

$$\mathcal{A}_N(2q) = 2 \sum_{\substack{l \le N \\ l \text{ odd}}} \sum_{j=1}^{N} s(j\overline{2q}, l) \tag{74}$$

$$+ 4\sum_{k=1}^{M}\sum_{r=1}^{M} s(r\bar{q}, k) + 2\sum_{k=1}^{M}\sum_{r=1}^{M} s((2r-1)\bar{q}, k).$$

The modulus-one terms are understood as zero. The bars denote inverses in the modulus of their summand. This formula follows by splitting (72) into odd and even $l$ and then using (73).

These three signed rectangles are the source that remains after the shift and parity reduction. Their uniform bound requires an analytic transfer beyond the unshifted unit-limit theorem.

26.5. **Parity transfer.** Parity restricts the source to a fixed number of arithmetic progressions in the reciprocity calculation. The following estimate makes that transfer explicit. Its analytic ingredients are the reciprocal-phase, completion and stopped-descent estimates in the original proof of [21, Sections 4–12]. That original proof remains the input to this parity transfer. The revised conductor estimate quoted in Section 25 concerns the unshifted source and is not substituted into the shifted argument.

The classical background includes Dedekind reciprocity [23] and estimates for complete exponential sums, including Kloosterman sums [8, Chapter 11]. The progression restrictions and stopped descent needed here are supplied by the cited sampling proof.

The two cutoff variables $X, Y$ describe a rectangle of indices, allowing its sides to end at different places. Uniformity in both is needed because splitting into even and odd indices changes those endpoints. The proof first converts clock correlations into modular interval counts. Finite Fourier sums retain the restriction to odd indices while completing the counts to whole residue systems. Reciprocity then reduces the modular pairs to smaller ones; the restriction is followed through that reduction. The required saving is below $p^2$. Once divided by the sampling scale $p^2$, the bound in the lemma tends to zero.

**Lemma 26.4** (Parity-restricted rectangles)**.** *Let $p$ be an odd prime, $1 \le X, Y < p$, and put $\eta = 1/458696$. Each of the following sums is bounded by*

$$O(p^{2-\eta}\log^4(2p)),$$

*uniformly in $X, Y$:*

$$2\sum_{l\le X}\sum_{j\le Y} s(j\overline{p^2},l),$$
$$2\sum_{\substack{l\le X\\ l \text{ odd}}}\sum_{j\le Y} s(j\overline{2p^2},l),$$
$$2\sum_{l\le X}\sum_{\substack{j\le Y\\ j \text{ odd}}} s(j\overline{p^2},l).$$

*Each inverse is taken in its summand's modulus; modulus-one terms are zero. The implied constant is absolute.*

*Proof.* Write $l = rk$, $j = ra$, $(a,k) = 1$, and set $U_r = \lfloor X/r\rfloor$, $L_r = \lfloor Y/r\rfloor$. The distribution identity for $\psi$ gives $s(j\bar{q},l) = s(a\bar{q},k)$. Pairing opposite units therefore expresses each rectangle as a sum of $s(h,k)W_{r,k}(qh)$ over $k \le U_r$ and $h \in \mathcal{U}_k$. Here $q = p^2$ or $2p^2$. For the first rectangle the source is

$$\Delta_L(y;k) = \mathbf{1}_{1\le y \bmod k\le L \bmod k} - \mathbf{1}_{1\le -y \bmod k\le L \bmod k}, \qquad L = L_r.$$

For the second it is the same source with $r, k$ both odd. For the third, $r$ is odd. At even $k$ every unit numerator is odd, so the source is again $\Delta_L$. At odd $k$ it is

$$W_L(y;k) = \#\{1\le a\le L : a \text{ odd},\ a\equiv y\ (k)\} - \#\{1\le a\le L : a \text{ odd},\ a\equiv -y\ (k)\}.$$

All these sources are odd and bounded by one in absolute value. The restriction on $r$ only removes terms from the absolute sums over source indices used below.

We first retain the endpoints of the new odd source. Put $R = \lceil L/2\rceil$, $m = R \bmod k$, $a_2 = (k+1)/2$, and $I_m(z) = \mathbf{1}_{1\le z \bmod k\le m}$. Counting $a = 2v-1$ gives

$$W_L(y;k) = I_m(a_2(y+1)) - I_m(a_2(1-y)). \tag{75}$$

For $b(x) = \{x\} - 1/2$, with its actual value $-1/2$ at integers,

$$I_m(z) = \frac{m}{k} - b((z-1)/k) + b((z-m-1)/k).$$

This identity holds for every integer $z$, including zero residues. The two constants $m/k$ cancel in (75). In its four $b$ terms, $m$ can be

replaced by $R$, since the difference is a multiple of $k$. On either parity class of $y$, the four phases have the form

$$\theta + \frac{\sigma y/2 + v}{k}, \qquad \sigma \in \{-1, 1\}, \quad v \in \{-1/2, -R - 1/2\}, \tag{76}$$

where $\theta = (\sigma y + 1)/2 \bmod 1$ is constant on that class. In particular $|v| \le p/2$. The ordinary source has the literal formula

$$\Delta_L(y; k) = \frac{1}{k} - 2b(y/k) + b((y - L - 1)/k) + b((y + L)/k)$$
$$(y \not\equiv 0 \pmod{k}).$$

Its shifts have absolute value at most $p$.

Completion at fixed $k$ also survives. For centered nonzero frequencies, $|\widehat{I}_m(n)| \ll 1/|n|$. Expanding the two interval functions in (75) gives Kloosterman sums with second frequencies $\pm n a_2 q$ and unit-modulus constant factors. Since $(a_2 q, k) = 1$, their gcd factors are the same as those for $n$. The zero frequency cancels. Thus, for a weight $f$ with $f(0) = 0$ and cyclic variation $V_k$, the completion estimate remains

$$\left| \sum_{c \in \mathcal{U}_k} f(c) \sum_{r \le X/k} W_{r,k}(q\bar{c}) \right| \ll \frac{pV_k}{k} \tau(k) \big( \sqrt{k} \log^2(2k) + \log(2k) \big). \tag{77}$$

Any restrictions on $r$ are retained. This argument uses the Fourier coefficients before the affine permutation. It does not assert small variation of $W_L$ in its original residue order.

Inversion and reflection now give the same nearest-remainder descent of $s(c, k)$ as for the ordinary prefix. A parity condition on $k$ is fixed during completion. The stopped-cost weight therefore keeps the bound $V_k \le 8(k/H)^2$ from the complete descent. The other estimates sum reciprocal phases along $k = ct + a$, or along $c = dt + a$, $k = mc \pm d$. In each case an integer representative of $\bar{c}$ is affine in $t$. For instance, with $fa \equiv 1 \pmod{c}$,

$$h = (1 - fk)/c$$

is such a representative in the first case. In the second one may use $h = (\varepsilon f k - \varepsilon m)/d$, where $fa \equiv 1 \pmod{d}$ and $k = mc + \varepsilon d$. The modulus and representative therefore have fixed parities on at most two progressions in $t$.

On each progression the phases in (76) are reciprocal phases with coefficient $q/2$ or $-q/2$; ordinary sources have coefficient $q$. Their magnitudes are comparable to $p^2/c^2$ or $p^2/d^2$. The endpoint shifts are at most $p/2$ for coefficient $p^2/2$, and at most $p$ for coefficients $p^2, 2p^2$. The strict inequalities $2c < k < p$ and $4d < k < p$ keep these shifts smaller than the leading amplitudes by fixed factors. Replacing $t$ by $2v+\epsilon$ in a reciprocal phase only changes its amplitude and denominator shift by fixed factors. The reciprocal-phase bound $O(|Z|^{1/3}\log(2p))$ and the variation estimates for its Dedekind weight therefore apply on every resulting interval.

With $\ell = \log(2p)$, the first small pairs, rational correction, first crossing and complete stopped cost consequently have bounds

$$\begin{aligned} p^{5/3}H^{1/3}\ell^3, \qquad & p^{5/2}H^{-1}\ell^3, \\ p^{5/3}H^{1/3}\ell^3 + pH\ell^4, \qquad & p^{7/2}H^{-2}\ell^3, \end{aligned} \tag{78}$$

respectively. These are the four estimates of the original descent. The preceding completion and progression checks account for each change to its source. Distinct cutoffs $U_r, L_r$ do not alter them: $U_r$ bounds the interval length, while $L_r$ supplies only the retained shifts, with the bounds just stated.

It remains to check the full terminal population, where a restriction cannot be discarded from a signed sum. Include the crossing step in its prefix matrix $\left(\begin{smallmatrix} A & B \\ C & E \end{smallmatrix}\right)$, with determinant $\delta$. Its terminal variables obey

$$u > H, \quad 1 \le e \le H, \quad 2e < u, \quad (u,e) = 1, \qquad k = Au + Be \le U_r.$$

Put $f = \bar{u} \bmod e$, $v_0 = (1 - fu)/e$, using $f = 0$ at $e = 1$, and $h = Av_0 - Bf$. The determinant identity gives $\bar{c} \equiv \delta h \pmod{k}$. Since every source is odd, $\delta W_{r,k}(q\bar{c}) = W_{r,k}(qh)$. The two orientations thus still join into the full prefix population

$$(A,B) = 1, \quad |B| \ge 2, \quad B/A \in (-\beta, \alpha),$$
$$\alpha = (\sqrt{5}-1)/2, \quad \beta = 1 - \alpha.$$

The parity of $k$ remains a condition on this population, and the parity of $h$ is used only to expand its already joined source. Let $G_{r,K}(u,e)$ be the sum of $W_{r,k}(qh)$ over this population with $K \le A < 2K$, the cutoff $k \le U_r$ and the retained modulus parity. The terminal block is $\mathcal{B}_r(K;H) = 2\sum_{u,e} s(e,u)G_{r,K}(u,e)$, over the terminal domain displayed above.

For a small dyadic block $K \le A < 2K$, set $u = et + a$ at fixed matrix and residue $a$. Both $k$ and $h = \delta(A(1 - fu)/e - Bf)$, before the orientation fold, are affine in $t$. Splitting its parity again costs a fixed number of progressions. The original small-block estimate therefore reads

$$|\mathcal{B}_r(K; H)| \ll p^{2/3} U_r H^{1/3} K \ell^3. \tag{79}$$

Here the Dedekind weight's endpoint size and variation are $O(U_r/(Ae) + 1 + |s(a, e)|)$, exactly as in the unmodified descent.

For a large block, hold $u, e, B$ fixed and remove coprimality by $\mathbf{1}_{(A,B)=1} = \sum_{d|A,\ d|B} \mu(d)$, $A = da$. The remaining inequalities give one interval in $A$. Since $k = Au + Be$ and $h = Av_0 - Bf$ are affine in $A$, all parity restrictions leave at most two progressions of step at most two in $a$. On the odd-modulus branch an even $d$ contributes nothing. The map $(A, B) \mapsto (k, h)$ has determinant $-1$, so $(k, h) = (A, B) < k$. Thus the ordinary literal formula still applies after coprimality is removed: $q$ is coprime to every allowed $k$. The affine interval formula needs no such exclusion and remains exact on these nonprimitive pairs as well.

For any of the retained phases put $q' = q$ or $q' = \pm q/2$, as appropriate. As a real identity,

$$\frac{q'h + v}{k} = \gamma - \frac{Z_v}{A + \zeta},$$

$$Z_v = \frac{q'B}{u^2} - \frac{v}{u}, \quad \zeta = \frac{eB}{u}, \quad \gamma = -\frac{q'f}{e} + \frac{q'}{eu}.$$

The same interval inequalities give $|Z_v| \asymp p^2|B|/u^2$ and $A + \zeta \asymp K$, with absolute comparison constants. In particular, the shift ratio is at most $2/(K|B|)$, since $u < 2p/K$ and $|B| \ge 2$. Take $H = \lfloor p^{11/14} \rfloor$, $K > p^{1/28}$, and $D = J = \lfloor K^{1/100} \rfloor$. On $d \le D$ put $M = K/d$. For frequencies $1 \le j \le J$, the phase derivatives on each parity progression have sizes comparable to

$$TM^{-a} \quad (2 \le a \le 14), \qquad T = j|Z_v|/K, \qquad K \ll T \ll K^{12+1/100}.$$

Thus $M^{1/2} \le T \le M^{13}$ once $K$ exceeds an absolute constant. The finitely many smaller dyadic scales are absorbed by increasing the implied constant. The fixed derivative-window estimate from the original proof of [21, Section 11] supplies the saving $\kappa = (2^{14} - 2)^{-1}$ on each progression. Erdős–Turán summation includes integer phase

hits and gives

$$O\big(M/J + M^{1-\kappa}\log(2J) + 1/(du)\big)$$

for each $B, d$; the last term retains the ordinary source's $1/k$. There are $O(K/d)$ possible multiples $B$ of $d$. Summing them and then bounding $d > D$ by its population gives

$$\begin{aligned}|G_{r,K}(u,e)| &\ll K^2/J + K^{2-\kappa}\log(2K)\\ &\quad + K/u + K^2/D + K\log(2K)\\ &\ll K^{2-\kappa}\log(2K).\end{aligned}$$

No parity restriction has been removed from the joined source.

The terminal Dedekind weights have total absolute mass $O((U_r/K)^2\ell^2)$. Use (79) below $K_0 = p^{1/28}$ and the preceding saving above it. The bounds $\sum_r U_r \ll p\ell$ and $\sum_r U_r^2 \ll p^2$ give a total terminal cost

$$O\big(p^{55/28}\ell^4 + p^{2-\kappa/28}\ell^3\big)\,.$$

At the chosen $H$, (78) costs $O(p^{27/14}\ell^3)$. Since $\eta = \kappa/28$, all contributions satisfy the asserted estimate. □

26.6. **Completion of the half-cell limit.** The parity estimate now supplies the signed bound required by Theorem 26.2. Each rectangle below is one of the terms retained in the shifted source.

*Proof of Theorem 26.2.* The three terms of (74) are respectively the second rectangle in Lemma 26.4, twice its first rectangle with $X = Y = M$, and its third with $X = M$, $Y = N$. Thus $\mathcal{A}_N(2p^2) = O(p^{2-\eta}\log^4(2p))$. The first rectangle with $X = Y = N$ supplies the same bound for $\mathcal{A}_N(p^2)$. Equation (71) gives that bound for $\mathcal{A}_p^{\mathrm{half}}$. Insert it in (69); $M^2/p^2 = 1/4 + O(1/p)$ and the harmonic term is $O(p^{-2}\log(2p))$. □

The half-cell account is complete. Its point contribution and signed remainder have both been followed under the changed observation. The next application changes the field as well as its cutoff, so its energy change will include a removed shell and inherited mixed work.

## 27. Completed cycles and weighted interaction

A field built from many periodic families presents two different scales. An old family may have completed its source profile and repeat with zero mean, while a new family is still changing. Their interaction is controlled by the old primitive and the variation of its weighted partner.

The question throughout this application is how much the complete native energy can increase when the scale doubles. We first specify the divisor source and the desired allowance. The weighted rules of Section 10 then control the interaction of short completed cycles with all changed families. The cycle length remains in that estimate, so its sum covers only part of the whole energy increment.

Here a return is a literal period of a fractional-clock field. The observation uses unequal native masses. A zero period mean therefore does not make the observed interaction vanish.

27.1. **Tapered divisor families.** Each odd label $r$ starts a family at length $r$ and follows it through $2r, 4r, 8r$. The profile assigns the signed levels on those successive bands. The taper $\tau_S$ keeps the profile unchanged through $S$, reduces it linearly to zero at $2S$, and vanishes beyond that point. Multiplying the tapered profile's first difference at $j$ by $j$ produces the clock coefficients. This order fixes the coefficients at every band boundary and at $2S$.

The families are joined with the Möbius coefficients introduced in Section 16. Their signs remain in the combined field before it is squared. The same divisor-inversion identity will later connect the family source to its reset readings.

Let $t \geq 1$ be an integer, $S = 4t$, and

$$\tau_S(j) = \min(1, \max(0, (2S - j)/S)).$$

For each positive odd core $r$, write its profile as

$$\phi_r(j) = \frac{1}{r}\left(\mathbf{1}_{\{r \leq j < 2r\}} - \mathbf{1}_{\{2r \leq j < 4r\}} + \frac{1}{4}\mathbf{1}_{\{4r \leq j < 8r\}}\right).$$

Define the family source, raw field and joined source by

$$q_{t,r}(j) = j\Delta[\tau_S(j)\phi_r(j)], \qquad f_{t,r}(n) = \sum_j q_{t,r}(j)\{n/j\},$$

$$q_t(j) = \sum_{r \text{ odd}} \mu(r) q_{t,r}(j), \qquad p_t(n) = \sum_{r \text{ odd}} \mu(r) f_{t,r}(n),$$

$$B_t = E_S[p_t].$$

Only $r < 2S$ occur in the parent, and every source includes its terminal coefficient at $j = 2S$. These are the prescribed Möbius families. Indeed, with $A(x) = \sum_{d \le x} \mu(d)/d$,

$$A(j) - \frac{3}{2}A(j/2) + \frac{1}{2}A(j/4) = \sum_{r \text{ odd}} \mu(r)\phi_r(j).$$

This follows by separating odd $d$ and $d = 2r$ in $A$; terms divisible by four have zero Möbius coefficient. Denoting the left side by $F(j)$ gives the original source $q_t(j) = j\Delta[\tau_S F](j)$. In particular $q_t(1) = 1$ is retained, and $\sum_j q_{t,r}(j)/j = \sum_j q_t(j)/j = 0$ by telescoping. The real-address fields are constant on open unit cells, so their continuous native integral in (18) uses exactly those clock readings.

At $t = r = 1$, Figure 45 follows the three operations on eight addresses. The last source coefficient records the final drop to zero. Omitting it would destroy reciprocal neutrality.

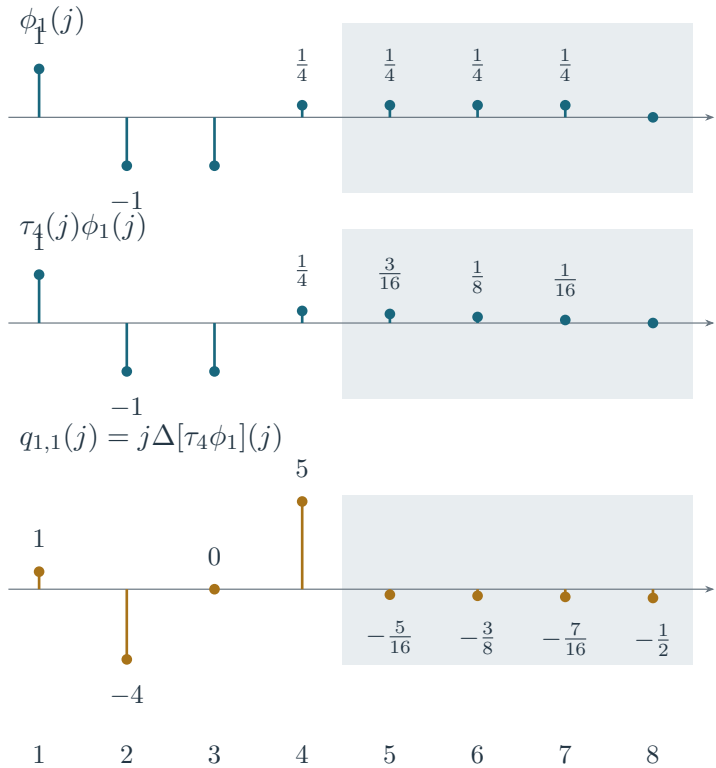



FIGURE 45. One family at $t = 1$, $S = 4$, $r = 1$. The first row is its profile; the second applies the taper; the third takes the weighted difference. The shaded addresses lie beyond $S$. The terminal coefficient $q_{1,1}(8) = -1/2$ is retained even though the tapered profile is zero there. Including it gives $\sum_j q_{1,1}(j)/j = 0$. Only integer readings are drawn.

At doubling put $\delta f_r = f_{2t,r} - f_{t,r}$ and $\delta p = p_{2t} - p_t$. Only $S/8 < r < 4S$ can change. If $8m \le S$, the complete old family is unchanged and equals

$$H_m(n) = \{n/m\} - 4\{n/(2m)\} + 5\{n/(4m)\} - 2\{n/(8m)\}.$$

Its eight-block profile is

$$H_m(n) = h(\lfloor n/m \rfloor \bmod 8), \qquad h = (0, -1, 2, 1, -1, -2, 1, 0).$$

27.2. **The complete increment.** For this prescribed source, the desired bound is

$$B_{2t} - B_t \le C\ell_t^3 \qquad (t \ge 1), \qquad \ell_t = 1 + \lfloor \log_2 t \rfloor, \tag{80}$$

with one constant $C$ independent of $t$. This is a one-sided bound on the whole energy change. Summed through successive doublings, it would control the accumulated growth by a fourth power of the number of scales. The estimate remains open.

The moving-field identity gives three contributions to this change. The old field loses its shell $S < n \le 2S$. On the common tail, the change interacts with the old field and contributes its own square. Screening will first bound a portion of that inherited interaction. Sections 28 and 29 then retain the complete change while examining its tail and shell.

27.3. **Family size and changed resets.**

**Lemma 27.1** (Family size and changed reset variation)**.** *For every positive odd $r$, $|f_{t,r}(n)| \le 6$ at all integer addresses. The difference source $\delta q_r = q_{2t,r} - q_{t,r}$ satisfies*

$$\delta q_r(j) = 0 \ (j \le S \ \text{or} \ j > 4S),$$
$$\sum_j |\delta q_r(j)| \le 20, \qquad \sum_j \frac{\delta q_r(j)}{j} = 0. \tag{81}$$

*Consequently $|\delta f_r| \le 12$ and $\Delta \delta f_r(n) = -\sum_{j|n} \delta q_r(j)$.*

*Proof.* Put $r_j(n) = n \bmod j$ and $Z_j = r_j - r_{j+1}$. Finite summation by parts in the source, retaining the zero terminal profile, gives

$$f_{t,r} = V_{S,r} - 2V_{S,2r} + V_{S,4r}, \qquad V_{S,a} = \frac{1}{a} \sum_{j=a}^{\min(2a-1,2S-1)} \tau_S(j) Z_j.$$

For a nonempty sum ending at $u$,

$$aV_{S,a} = \tau_S(a) r_a + \sum_{j=a+1}^{u} (\tau_S(j) - \tau_S(j-1)) r_j - \tau_S(u) r_{u+1}.$$

After the first term the coefficients are nonpositive and their absolute sum is $\tau_S(a)$. The corresponding residues are less than $2a$, and $r_a < a$. Thus $-2\tau_S(a) \le V_{S,a} \le \tau_S(a)$, also for an empty sum. The signed combination $V_{S,r} - 2V_{S,2r} + V_{S,4r}$ lies between $-6$ and 6.

For the difference source put $\theta_S = \tau_{2S} - \tau_S$. It rises from zero at $S$ to one at $2S$ and falls to zero at $4S$, with total discrete variation two. The product difference is exactly

$$\delta q_r(j) = \theta_S(j)\nu_r(j) + j\Delta\theta_S(j)\phi_r(j-1),$$

where $\nu_r = j\Delta\phi_r$ takes values $(1, -4, 5, -2)$ at $(r, 2r, 4r, 8r)$. Its absolute mass is twelve, and $j|\phi_r(j-1)| \le 4$. Therefore the difference mass is at most $12 + 4 \cdot 2 = 20$. The support and reciprocal identity follow from the product $\theta_S\phi_r$, including its terminal difference at $4S$. The field difference bound follows from the two family bounds; its reset identity follows from reciprocal neutrality. □

27.4. **Completed-cycle screening.** A completed old family repeats with zero mean. Against a constant partner, each complete period contributes zero. A changing weighted partner leaves work controlled by its variation, just as the alternating four-term example did in Section 8. Here the period has length $8m$, so the primitive bound must retain its dependence on $m$. That dependence determines how many families the estimate can control.

The completed profile has zero period mean and $|H_m| \le 2$. Its primitive $F_m(N) = \sum_{n=1}^{N} H_m(n)$ stays between $-m$ and $2m$. To see this, the values at block ends $N = jm - 1$, $j = 1, \dots, 8$, are $0, -m, m, 2m, m, -m, 0, 0$. Between block ends the increments have constant sign, and complete periods sum to zero. Subtracting two primitive values consequently gives

$$\left| \sum_{n=a}^{b} H_m(n) \right| \le 3m \qquad (1 \le a \le b). \tag{82}$$

The field and primitive for $m = 1$ appear in Figure 46. For general $m$, the block lengths and primitive range grow with $m$; that dependence remains in the screening estimate.

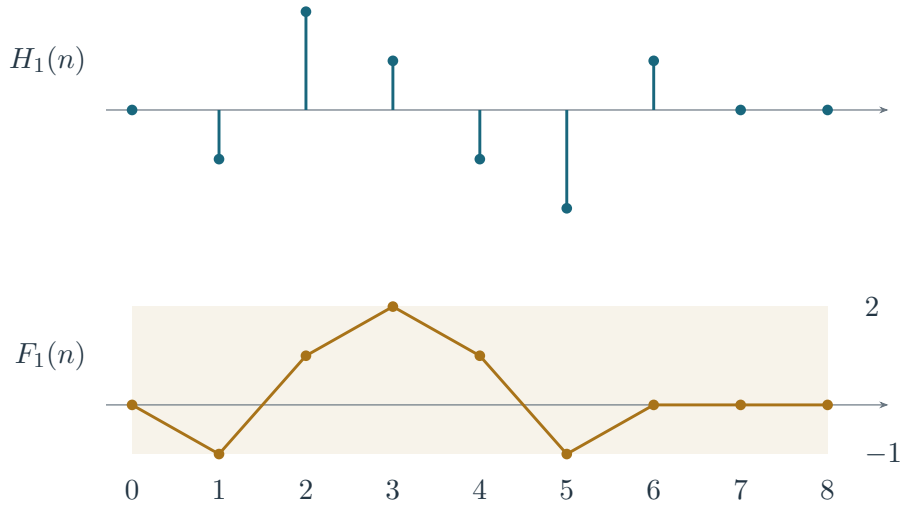



FIGURE 46. One completed period and its anchored primitive. The field sums to zero over the period, while the primitive stays in $[-1, 2]$. Any interval sum is a difference of two primitive values and is at most three in magnitude. The weighted sum also depends on variation of the native masses.

**Theorem 27.2** (Completed-family screening)**.** *Let $S = 4t$, $t \geq 1$, let $m$ be odd with $8m \leq S$, and let $r$ be any positive odd core. Then the complete interaction*

$$K_S(m, r) = \sum_{n>2S} w_n H_m(n) \delta f_r(n)$$

*obeys*

$$|K_S(m, r)| \leq 129 \frac{m}{S^2}. \tag{83}$$

*Proof.* Use Proposition 10.2 with $a = 2S + 1$, $u = H_m$ and $v = \delta f_r$. The primitive bound is $3m$ and $|v| \leq 12$. By the complete signed reset identity,

$$\sum_{n \geq a} w_{n+1} |\Delta v(n+1)| \leq \sum_j |\delta q_r(j)| \sum_{\substack{k \geq a+1 \\ j|k}} w_k$$

$$\leq 20 \frac{2}{S^2}.$$

For the last inequality, all changed clocks have $j > S$, and

$$\sum_{\ell \ge 1} w_{j\ell} \le \frac{1}{j^2} \sum_{\ell \ge 1} \frac{1}{\ell^2} < \frac{2}{j^2} \le \frac{2}{S^2}.$$

The reciprocal-square sum is less than two, since

$$\frac{1}{\ell^2} < \frac{1}{\ell - 1} - \frac{1}{\ell} \qquad (\ell \ge 2).$$

The contribution from varying weights is $12 w_a \le 3/S^2$. Hence (22) gives

$$|K_S(m, r)| \le 3m(3 + 40)/S^2 = 129m/S^2.$$

Its convergence hypotheses are verified by these same estimates, and the upper endpoint tends to zero. No period mean of the changed family has been subtracted. □

The bound uses a small primitive of the old field, not a universal sign of its interaction. The infinite tail and the actual jumps of its partner remain present throughout.

**Problem 52** (A completed cycle under native weights)**.** Use the completed profile $H_1(n) = h(n \bmod 8)$ with

$$h = (0, -1, 2, 1, -1, -2, 1, 0).$$

Compute its primitive at addresses $0, 1, \dots, 8$. Find an interval attaining the bound $|\sum_{n=a}^{b} H_1(n)| \le 3$. Compare its unweighted and native weighted sums over $1 \le n \le 8$. Explain which additional variation must be retained when a zero-mean cycle is observed with native weights.

27.5. **Summed screening bound.** For an integer $0 \le M \le S/8$, define the short old field and its work

$$p_{\le M}(n) = \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) H_m(n), \qquad W_S(M) = 2 \sum_{n > 2S} w_n p_{\le M}(n) \delta p(n).$$

The core sums are finite and the native products are absolutely summable, so

$$W_S(M) = 2 \sum_{\substack{m \le M \\ m \text{ odd}}} \sum_{\substack{S/8 < r < 4S \\ r \text{ odd}}} \mu(m) \mu(r) K_S(m, r).$$

There are at most $2S$ changed odd labels. Since the odd positive integers through $M$ sum to $\lceil M/2\rceil^2$, the theorem gives

$$|W_S(M)| \le \frac{516}{S}\lceil M/2\rceil^2 \le 129\frac{(M+1)^2}{S}. \tag{84}$$

For

$$M_S = \min(\lfloor S/8\rfloor, \lfloor\sqrt{S}\rfloor)$$

this is at most 516. It controls the interaction of all those short old cycles with every changed family. No cancellation among their Möbius signs is required for this portion.

The same proved estimate also allows a logarithmically longer range. With $\ell_t = 1 + \lfloor \log_2 t\rfloor$, take

$$M_t^\star = \min\bigl(\lfloor S/8\rfloor, \lfloor\sqrt{S\ell_t^3}\rfloor\bigr).$$

Then (84) gives $|W_S(M_t^\star)| \le 516\ell_t^3$. This controls the complete work against every changed family through that cutoff. A range $S^{1/2+\epsilon}$ for fixed $\epsilon > 0$ would require a further saving.

Taking $M = \lfloor S/8\rfloor$ in the same bound instead gives $O(S)$. The number and total cycle scale of the old families consume the saving in the pair estimate. This is an upper allowance from that estimate, not a lower bound on the actual signed work.

27.6. **Finite evaluation of an infinite interaction.** The screening bound concerns the complete series. A finite calculation can approximate it with an explicit error. For $N \ge 2S$ set

$$K_{S,N}(m,r) = \sum_{2S<n\le N} w_n H_m(n)\delta f_r(n).$$

The bounds $|H_m| \le 2$ and $|\delta f_r| \le 12$ give

$$|K_S(m,r) - K_{S,N}(m,r)| \le 24\sum_{n>N}\frac{1}{n(n+1)} = \frac{24}{N+1}.$$

Thus the finite rational sum specifies an interval containing the infinite interaction. Taking $N \ge 2399$ makes its half-width at most $1/100$; a valid cutoff must also satisfy $N \ge 2S$. The calculation requires the actual fractional-clock readings at each integer, including resets. An equivalent evaluation uses the neutral floor sum from Section 10.

The finite interval and the all-scale theorem answer different questions. The interval encloses one chosen interaction; the screening theorem controls every admissible pair with its stated dependence on $m, S$.

27.7. **Large primitives and the screening bound.** Joining the old families before estimating gives a more informative version of the same method. Put

$$F_M(n) = \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) F_m(n), \qquad J_{S,M} = \sup_{n \ge 2S} |F_M(n) - F_M(2S)|.$$

The weighted variation of the complete changed field is at most $86/S$. Indeed there are at most $2S$ changed labels, and the proof of Theorem 27.2 bounds each weighted variation bracket by $43/S^2$. Therefore

$$|W_S(M)| \le 172 J_{S,M}/S. \tag{85}$$

The signs have now entered $F_M$. They do not, however, make this primitive uniformly small at every address.

**Proposition 27.3** (Large primitives of the actual old source)**.** *For every integer $M \ge 10$ and every integer anchor $a \ge 0$,*

$$\sup_{n \ge a} |F_M(n) - F_M(a)| \ge \frac{1}{2} \sum_{\substack{M/2 < p \le M \\ p \text{ prime}}} (3p - 12).$$

*Consequently this supremum is $\gg M^2/\log M$ as $M \to \infty$, uniformly in the anchor.*

*Proof.* Let $\mathcal{P}$ be the primes in $(M/2, M]$. Each is at least seven and divides no odd label $m \le M$ other than itself. Let $L$ be the least common multiple of all odd labels through $M$ outside $\mathcal{P}$, with the empty least common multiple equal to one. Every $p \in \mathcal{P}$ is coprime to $8L$ and to the other selected primes.

Restrict addresses to $n \equiv 3 \pmod{8L}$. All unselected primitives then have fixed readings. For a selected prime, $F_p(4p-1) = 2p$. Put $d_p = 2$ if $p \equiv 1 \pmod 4$ and $d_p = 6$ if $p \equiv 3 \pmod 4$. Both $4p-1$ and $6p-1-d_p$ are congruent to three modulo eight. The second address lies in the block with increment $-2$, so $F_p(6p-1-d_p) = -p + 2d_p$.

The Chinese remainder theorem allows the residue modulo each selected $p$ to be chosen independently while retaining $n \equiv 3 \pmod{8L}$.

Choose one address with every selected primitive at its first reading, and another with every selected primitive at its second reading. Since $\mu(p) = -1$, the second joined reading minus the first is

$$\sum_{p\in\mathcal{P}}(3p - 2d_p) \geq \sum_{p\in\mathcal{P}}(3p - 12).$$

Both addresses can be moved beyond any anchor by adding their common period. At least one reading is at distance half this difference from $F_M(a)$. This proves the finite inequality. Finally the prime number theorem [24] gives $\sum_{M/2<p\leq M} p \sim 3M^2/(8\log M)$, while the number of selected primes is $O(M/\log M)$. The asserted growth follows. □

For $M$ of order $S^{1/2+\epsilon}$, any fixed $\epsilon > 0$, the uniform primitive estimate required to make (85) logarithmic is therefore false. The large readings may occur at very distant addresses. Their existence does not give a large native weighted interaction, because those addresses have small masses. A useful refinement must retain the weighted location of the primitive and its correlation with the changed source.

**Problem 53** (The range of a summed screening estimate)**.** Starting with $|K_S(m, r)| \leq 129m/S^2$, use at most $2S$ changed odd labels to recover an allowance for $W_S(M)$, with $M \leq S/8$. At $S = 256$, evaluate that allowance for $M = 16$ and $M = 32$. For $M = \lfloor S^\alpha \rfloor$, $0 < \alpha < 1$, determine its order in $S$ and the range of $\alpha$ for which it stays bounded. Identify the loss when every completed cycle is included. Does this calculation prove that the actual signed work grows at that order?

## 28. Native energy under refinement

At a doubled scale, both the clock field and its activation cutoff change. The observation loses the parent shell. On the common tail, the child field replaces the parent. The field-change identity of Section 10 determines the whole increment before any families are separated.

Retain the fields of Section 27, with $S = 4t$ and $B_t = E_S[p_t]$. We first localize the full increment to a finite interval with a controlled tail. Aligning completed cycles then exposes the parent-shell credit,

and combining resets gives a finite formula for the remaining work. Each step preserves the same energy change.

28.1. **Complete energy change.** Put $p_> = p_t - p_{\le M_S}$. The full moving-field identity is

$$B_{2t} - B_t = W_S(M_S) + R_S^{\mathrm{nat}}, \tag{86}$$
$$R_S^{\mathrm{nat}} = -\sum_{S<n\le 2S} w_n p_t(n)^2 + \sum_{n>2S} w_n\big(2p_>(n)\delta p(n) + \delta p(n)^2\big).$$

The negative parent shell has not been split. The common tail includes all interactions with the remaining old cores and every interaction between changed cores. In particular $p_>$ contains intermediate completed cycles as well as families near the moving cutoff.

The scale count $\ell_t = 1 + \lfloor \log_2 t \rfloor$ is the one in the upper-bound question posed in Section 27. The screened term is already bounded. The rest of the calculation must control $R_S^{\mathrm{nat}}$ with all three contributions retained.

28.2. **Finite localization of native work.** The weight of an address now supplies an effective limit on how far large primitive readings can contribute significant work. There are at most $S$ odd labels in the parent and $2S$ in the child. Lemma 27.1 therefore gives, at every integer address,

$$|p_t(n)| \le 6S, \qquad |p_{2t}(n)| \le 12S.$$

These bounds are coarse in amplitude but useful in the distant tail.

**Corollary 28.1** (Quadratic cutoff)**.** *Let $S = 4t$, $t \ge 1$, $Y = p_{2t} - p_t$, and $N \ge 2S$ be an integer. Define*

$$H_t(N) = -\sum_{S<n\le 2S} w_n p_t(n)^2 + \sum_{2S<n\le N} w_n\big(2p_t(n)Y(n) + Y(n)^2\big). \tag{87}$$

*Then*

$$-\frac{36S^2}{N+1} \le B_{2t} - B_t - H_t(N) \le \frac{144S^2}{N+1}. \tag{88}$$

*In particular, put* $H_t = H_t(S^2)$. *The assertions*

$$B_{2t} - B_t \le C\ell_t^3 \quad (t \ge 1), \qquad H_t \le C'\ell_t^3 \quad (t \ge 1)$$

*are equivalent with possibly different absolute constants.*

*Proof.* Apply Theorem 10.4 with the two displayed field envelopes and native weights. Since $S \ge 4$, the cutoff $S^2$ is beyond $2S$. At that cutoff the omitted change lies strictly between $-36$ and $144$. Thus an upper bound $C'\ell_t^3$ for $H_t$ gives $(C'+144)\ell_t^3$ for the complete increment. Conversely an upper bound $C\ell_t^3$ for the increment gives $(C+36)\ell_t^3$ for $H_t$, because $\ell_t \ge 1$. □

The infinite tail has now been accounted for by an absolute allowance. The remaining arithmetic estimate concerns the finite addresses $S < n \le S^2$. A large primitive reading obtained by a distant congruence construction need not occur inside this interval. Localization retains that distinction without requiring a uniform bound on the primitive over all addresses.

Figure 47 shows the whole observation after localization. The finite head retains the shell and signed tail; the two envelopes pay for every omitted address.

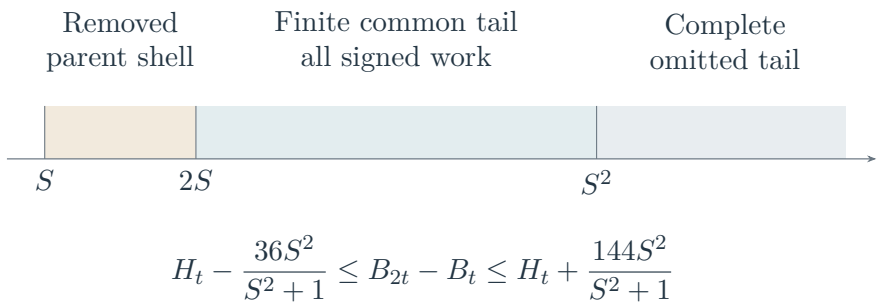



FIGURE 47. The quadratic cutoff keeps the parent shell and the entire finite tail in one calculation. The two field envelopes control all later addresses. The displayed interval contains the complete increment at every scale. Horizontal lengths are schematic; the cutoffs and allowances are exact.

28.3. **Cycle alignment and parent credit.** The common start at $2S$ can cut an old cycle in the middle. Moving that family's start

back to its previous period boundary makes its primitive start at zero. But the move restores readings between the two starts. Those fragments all lie in the parent shell already removed by the activation change. Their combined field $Z_S$ must therefore be accounted for together with the old shell field. The construction below keeps that adjustment as a signed difference of two whole squares.

Here take $M = \lfloor S/8 \rfloor$, so every completed old family is included. The earlier cutoff $M_S$ selected only the sufficiently short cycles controlled by screening. The alignment below is an identity for the larger range; estimating its signed work remains a separate task. Put $Y = \delta p$ and define

$$L_m = 8m \left\lfloor \frac{2S}{8m} \right\rfloor, \qquad Z_S(n) = \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) H_m(n) \mathbf{1}_{\{L_m < n \le 2S\}},$$

$$C_M = p_{\le M}, \qquad P_> = p_t - C_M,$$

$$\widetilde{W}_S = 2 \sum_{\substack{m \le M \\ m \text{ odd}}} \mu(m) \sum_{n > L_m} w_n H_m(n) Y(n).$$

Since $8m \le S$, one has $S < L_m \le 2S$. Every restored fragment lies inside the same removed parent shell.

For $S = 28$, the completed family $m = 3$ has period twenty-four. Its last period boundary before $2S = 56$ is $L_3 = 48$. Moving the start back restores the readings at $49, \dots, 56$, as shown in Figure 48. The correction is a field on the removed shell, so its interaction with the retained field must be included.

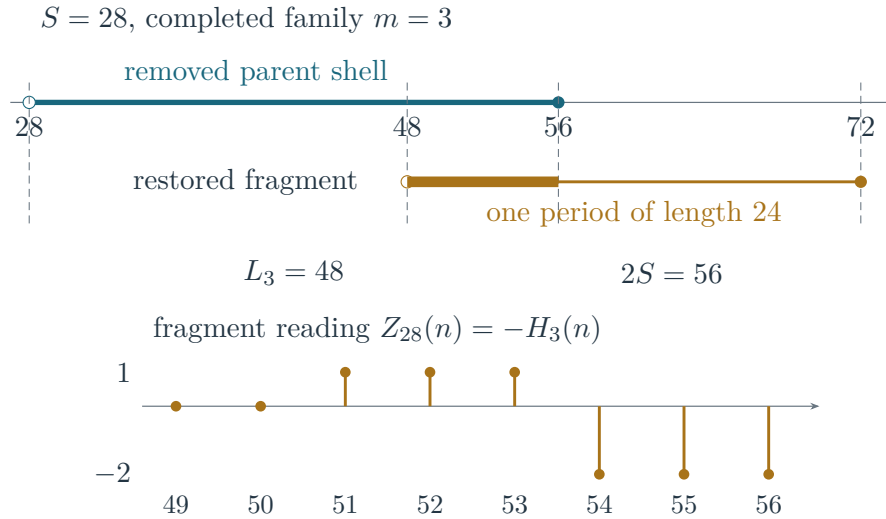



FIGURE 48. Moving the start from 56 back to 48 restores the eight integer addresses $49, \dots, 56$, all inside the removed shell. The other completed odd family, $m = 1$, already ends a period at 56 and restores nothing. Thus this example has one fragment with coefficient $\mu(3) = -1$. With several fragments, their fields must be added before squaring.

**Proposition 28.2** (Cycle alignment and the parent credit)**.** *For the prescribed divisor source,*

$$B_{2t} - B_t = T_S + \widetilde{W}_S + \sum_{n>2S} w_n\big(2P_>(n)Y(n) + Y(n)^2\big), \tag{89}$$

$$T_S = \sum_{S<n\le 2S} w_n\big(Z_S(n)^2 - (p_t(n) - Z_S(n))^2\big).$$

*All series converge absolutely at each fixed scale. The sum of individual fragment energies satisfies*

$$D_Z = \sum_{\substack{m\le M\\ m\ \text{odd}}} \mu(m)^2 \sum_{L_m<n\le 2S} w_n H_m(n)^2 < \frac{1}{4}.$$

*Proof.* With $F(j) = A(j) - \frac{3}{2}A(j/2) + \frac{1}{2}A(j/4)$ as above, its untapered source is

$$\nu(j) = j\Delta F(j) = \mu(j) - 3\mathbf{1}_{2|j}\mu(j/2) + 2\mathbf{1}_{4|j}\mu(j/4).$$

The divisor sum of $\nu$ is $\mathbf{1}_{n=1} - 3\mathbf{1}_{n=2} + 2\mathbf{1}_{n=4}$. Reciprocal neutrality and the right-reading reset identity imply $p_t(1) = -1$, $p_t(2) = p_t(3) = 2$, and $p_t(n) = 0$ for $4 \le n \le S$. Applying the same argument to the child gives $p_{2t} = 0$ on $S < n \le 2S$, hence $Y = -p_t$ there.

Restoring each fragment changes the completed old work by

$$\widetilde{W}_S - W_S(M) = 2 \sum_{S<n\le 2S} w_n Z_S(n) Y(n) = -2 \sum_{S<n\le 2S} w_n Z_S(n) p_t(n).$$

Join this correction with the single negative parent-shell term in Proposition 10.3. Completing the square gives $-p_t^2 + 2Z_S p_t = Z_S^2 - (p_t - Z_S)^2$, proving (89). All fields are bounded at a fixed scale and the weights are summable.

Finally $|H_m| \le 2$ and there are at most $M$ selected labels. Since

$$\sum_{S<n\le 2S} w_n < 1/(2S),$$

we have $D_Z < 2M/S \le 1/4$ when $M > 0$. For $M = 0$ the sum is zero. □

The fragment diagonal is paid uniformly. The joined fragment square also contains all interactions between distinct families, so $D_Z$ cannot replace $\sum w_n Z_S(n)^2$. The useful quantity in the identity is the signed difference $T_S$ itself.

For the aligned work the primitive $F_m$ vanishes at $L_m$. The full infinite summation-by-parts identity is

$$\sum_{n>L_m} w_n H_m(n) Y(n) = \sum_{n>L_m} F_m(n)(w_n - w_{n+1}) Y(n) + \sum_j \delta q(j) \sum_{j\ell \ge L_m+2} w_{j\ell} F_m(j\ell - 1).$$

At a finite upper endpoint $N$, the first sum stops at $N-1$, the reset sum at $j\ell \le N$, and $F_m(N) w_N Y(N)$ must be added. The endpoint vanishes as $N \to \infty$. Weight variation is summable, and each of the finitely many reset progressions has summable weights. This justifies the displayed limit. Moving the initial boundary changes the reset cutoff as well as the primitive's initial value.

28.4. **Parent-shell response.** On the removed shell the actual divisor identity gives a shorter formula for the parent field. Retain the notation $F(n) = A(n) - \frac{3}{2}A(n/2) + \frac{1}{2}A(n/4)$.

**Lemma 28.3** (Parent response before the second cutoff)**.** *For $S < n \le 2S$, including the terminal address,*

$$p_t(n) = \frac{n(n-S)}{S}F(n) - \frac{1}{S}\sum_{k=S}^{n-1}(k-S)F(k). \tag{90}$$

*Proof.* Put $\nu(n) = n\Delta F(n)$. The divisor identity used in Proposition 28.2 gives $\sum_{j|n}\nu(j) = 0$ for $n > 4$. Every proper divisor of an address $S < n \le 2S$ is at most $S$, where $q_t(j) = \nu(j)$. The actual right-hand reset therefore satisfies

$$\Delta p_t(n) = \nu(n) - q_t(n) = \frac{n-S}{S}\nu(n) + \frac{n}{S}F(n-1).$$

The product difference for $\tau_S F$ gives the second equality, including $n = 2S$. The right side of (90) is zero at $n = S$. Its first difference is the displayed expression, and $p_t(S) = 0$. Induction proves the formula. □

This is a consequence of the prescribed Möbius source. Replacing its family coefficients arbitrarily would remove the divisor identity on which the proof depends. The formula evaluates the negative shell in (87); that shell remains the square of the whole parent field.

The first-shell formula evaluates the parent at every admitted address, including those at which restored cycle fragments are active. In the cycle credit, the later retained-field square contributes to the same account as the early shell. Keeping both joined squares preserves the signed difference in (89).

28.5. **Joined resets on doubling blocks.** After localization, the remaining task is a finite sum of energy changes. Grouping addresses into doubling blocks makes it possible to compare the sizes of contributions at different distances. The boundary at the end of one block is the start of the next, so its endpoint term must cancel when the blocks are rejoined. Within a block we keep the combined reset amplitude and its interaction with the preceding field.

The finite interval can be partitioned at $2S, 4S, 8S, \ldots$, ending with $S^2$. To evaluate its work by resets, combine the clocks that reset at

the same address. Define

$$a(n) = \sum_{\substack{j|n \\ j\le 2S}} q_t(j), \qquad d(n) = \sum_{\substack{j|n \\ j\le 4S}} \delta q(j), \qquad g(n) = p_{2t}(n)^2 - p_t(n)^2.$$

The source has been extended by zero outside its support. Reciprocal neutrality gives $\Delta p_t(n) = -a(n)$ and $\Delta Y(n) = -d(n)$.

**Proposition 28.4** (Finite signed reset work)**.** *For the actual source, put $J(n) = g(n) - g(n-1)$. Then*

$$\begin{aligned} J(n) = &-2Y(n-1)a(n) - 2\big(p_t(n-1) + Y(n-1)\big)d(n) \\ &+ 2a(n)d(n) + d(n)^2. \end{aligned} \tag{91}$$

*For any integers $2S \le u < v$,*

$$\sum_{u<n\le v} w_n g(n) = \frac{g(u)}{u+1} - \frac{g(v)}{v+1} + \sum_{n=u+1}^{v} \frac{J(n)}{n}. \tag{92}$$

*Consequently the complete finite head is*

$$H_t = -\sum_{S<n\le 2S} w_n p_t(n)^2 - \frac{p_t(2S)^2}{2S+1} - \frac{g(S^2)}{S^2+1} + \sum_{n=2S+1}^{S^2} \frac{J(n)}{n}. \tag{93}$$

*Proof.* At a reset, the old field changes from $p_t(n-1)$ to $p_t(n-1) - a(n)$, and the child changes from $p_t(n-1) + Y(n-1)$ to that value minus $a(n) + d(n)$. Subtract the two square changes. The old $a(n)^2$ cancels, leaving (91). Its mixed terms retain both the inherited field and the interaction between the two combined amplitudes.

For the weighted identity write $g(n) = g(u) + \sum_{k=u+1}^{n} J(k)$ and interchange the finite sums. The inner weight sum is $\sum_{n=k}^{v} w_n = 1/k - 1/(v+1)$. Collecting the coefficients of $g(u)$ and $\sum J = g(v) - g(u)$ gives (92).

Apply this identity on the consecutive doubling blocks. Each interior term $-g(v)/(v+1)$ cancels the next block's $g(u)/(u+1)$. The remaining initial value is $g(2S) = -p_t(2S)^2$, since the child vanishes there. Add the single parent shell to obtain (93). □

No intermediate block endpoint has acquired a separate charge. Formula (91) also retains interactions between distinct divisors at the same address. Splitting $a$ and $d$ into individual clock magnitudes would lose those signs.

The immediate envelope estimate on a block is

$$\sum_{u<n\le v} w_n g(n) \le 144S^2 \left( \frac{1}{u+1} - \frac{1}{v+1} \right).$$

Its sum from $2S$ to $S^2$ is of order $S$. Thus localization and regrouping have identified a finite signed target, but the desired logarithmic estimate still requires a bound for the actual correlations in (91). The parent shell and all block signs remain available together in (93).

**Problem 54** (Two-clock refinement and a complete tail)**.** Use the two-clock model $p(n) = \{n\} - 2\{n/2\}$ at integer addresses, with its actual right-hand readings. Replace it by $q(n) = p(n) + 1$ and move the native cutoff from two to four. Compute $E_4[q] - E_2[p]$ by separating the removed parent shell, inherited interaction and new square. Evaluate the infinite tails. Then form the finite head through $n = 10$ and use $|p|, |q| \le 1$ to enclose the omitted change. Decide its sign from that finite enclosure alone.

The finite reset formula keeps the removed shell in the same account as the common tail. The next section returns to that shell and asks how much of its retained energy later families can remove. A lower bound there contributes an upper bound to the increment.

## 29. Invariant readings on the parent shell

The parent-shell credit contains a negative retained-field square. A lower bound for that square can therefore give an upper bound for the energy change. This applies the invariant-reading method of Section 9 to a concrete family of allowed additions.

The later families vanish on the inherited prefix. On the parent shell, their first nonzero readings form a triangular matrix. That structure lets us construct local invariants and compare the information they retain with the full constrained minimum. A finite counterexample will show exactly what a sparse collection of readings can miss.

29.1. **Four-reading invariants.** The retained field in the parent credit is $p_t - Z_S$. The inherited prefix fixes the coefficients of the old families. Indeed, at an odd address $m \le S$ the untapered reset

amplitude is $\sum_{r|m} c_r$. The prescribed prefix makes it one at $m = 1$ and zero at all other odd $m \le S$; divisor inversion gives $c_r = \mu(r)$ for odd $r \le S$. Every odd family with $S < r < 2S$ vanishes through address $S$. These later families have a particularly simple response on the shell $I_S = \{S+1, \dots, 2S\}$. The following construction develops the invariant readings fixed by that prefix.

**Lemma 29.1** (Later family on the parent shell)**.** *For odd $S < r < 2S$ and integer $1 \le n \le 2S$,*

$$f_{t,r}(n) = \begin{cases} 0, & n < r, \\ \dfrac{n(n+1) + r(r-1) - 4Sr}{2Sr}, & r \le n \le 2S. \end{cases}$$

*The $S/2$ later families are linearly independent as vectors on $I_S$.*

*Proof.* Reciprocal neutrality gives $f_{t,r}(n) = -\sum_j q_{t,r}(j) \lfloor n/j \rfloor$. For $r > S$ the source is supported on $r \le j \le 2S$, and $n \le 2S < 2r$. Each nonzero floor is therefore one. The field is zero before $r$, and otherwise it is

$$-\sum_{j=r}^{n} j \Delta \left[ \frac{2S - j}{Sr} \right],$$

where the profile is zero before $r$ and its terminal value at $2S$ is zero. Summation by parts gives the displayed formula. In particular $f_{t,r}(r) = -(2S - r)/S \neq 0$. At the ordered entrance addresses the evaluation matrix is triangular with nonzero diagonal, proving independence. □

Write the inherited retained field and an arbitrary later addition as

$$v_S = \sum_{\substack{r \le S \\ r \text{ odd}}} \mu(r) f_{t,r} - Z_S, \qquad Ux = \sum_{\substack{S < r < 2S \\ r \text{ odd}}} x_r f_{t,r}.$$

The actual retained field is obtained by taking $x_r = \mu(r)$. Use the native shell norm $\|v\|_w^2 = \sum_{n \in I_S} w_n |v(n)|^2$.

The construction now seeks a combination of four readings that every later addition leaves unchanged. On a four-address block beginning at an odd address, families already active lie in the span of the two profiles 1 and $n(n+1)$. Only one additional odd family can enter strictly inside the block. Annihilating those three contributions gives three homogeneous linear conditions on four coefficients.

The following choice solves them; the proof checks that the resulting reading is nonzero.

Take four consecutive addresses beginning at an odd $q > S$, with $q + 3 \le 2S$. Put $G(n) = n(n + 1)$ and $s = q + 2$. Define

$$a_2 = f_{t,s}(s + 1), \qquad a_3 = -f_{t,s}(s),$$
$$a_1 = -\frac{[G(q + 2) - G(q)]a_2 + [G(q + 3) - G(q)]a_3}{G(q + 1) - G(q)},$$
$$a_0 = -a_1 - a_2 - a_3.$$

The reading on this block is $L_q(v) = \sum_{i=0}^{3} a_i v(q + i)$.

**Proposition 29.2** (Inherited four-reading bound)**.** *Each $L_q$ is nonzero and satisfies $L_q(Ux) = 0$ for every later coefficient vector $x$. For the disjoint blocks $q = S + 1, S + 5, \dots, 2S - 3$, put*

$$\mathcal{L}_S = \sum_q \frac{|L_q(v_S)|^2}{\sum_{i=0}^{3} a_i^2 / w_{q+i}}. \tag{94}$$

*Then, at every scale and for every $x$,*

$$\|v_S + Ux\|_w^2 \ge \mathcal{L}_S. \tag{95}$$

*Proof.* The coefficients satisfy $\sum_i a_i = 0$ and $\sum_i a_i G(q + i) = 0$. A later family entering at or before $q$ is a linear combination of these two modes throughout the block. The only later entrance strictly inside the block is $s = q + 2$; the definitions of $a_2, a_3$ give $a_2 f_{t,s}(s) + a_3 f_{t,s}(s + 1) = 0$. Families entering after the block vanish on it. This proves the annihilation assertion for every later family, hence for their whole span. Moreover $a_3 = (2S - s)/S > 0$, so the functional is nonzero. Theorem 9.3 applied to the disjoint blocks gives (95). □

For example, at $S = 4$ the only block is $\{5, 6, 7, 8\}$. Multiplying its coefficients by 84 gives the simpler reading

$$L(v) = 56v(5) - 80v(6) + 3v(7) + 21v(8).$$

It annihilates both later families, of cores five and seven. The reciprocal-weight denominator is

$$56^2(5 \cdot 6) + 80^2(6 \cdot 7) + 3^2(7 \cdot 8) + 21^2(8 \cdot 9) = 395136.$$

Thus any inherited field $v$ on this block obeys $\|v + Ux\|_w^2 \geq |L(v)|^2/395136$.

29.2. **Local bounds and the constrained minimum.** An invariant reading measures some of the energy that later additions cannot remove. A collection of readings can miss part of that energy. The comparison below tests whether the particular four-reading collection captures enough to pay for the restored fragments. Its failure will therefore test the chosen collection, not erase the retained energy.

The four-reading theorem gives an energy lower bound. To bound the parent adjustment by its fragment diagonal, a sufficient comparison would be

$$\|Z_S\|_w^2 - D_Z \leq \mathcal{L}_S. \tag{96}$$

Together with (95), it would imply $T_S \leq D_Z < 1/4$. This particular comparison fails for the actual source.

At $t = 51$, hence $S = 204$, exact rational evaluation gives

$$\|Z_S\|_w^2 - D_Z > \frac{3}{1000}, \qquad \mathcal{L}_S < \frac{1}{1000}.$$

At the same scale,

$$\|Z_S\|_w^2 < \frac{1}{100}, \qquad \|p_t - Z_S\|_w^2 > \frac{7}{100},$$

so the full parent adjustment is negative. The local readings detect only part of the retained energy. The same loss occurs at the scales shown below; decimals are rounded displays of exact rational values.

| $t$ | Fragment excess | $\mathcal{L}_S$ | $\mathcal{L}_S - (\|Z_S\|_w^2 - D_Z)$ |
|---|---|---|---|
| 51 | 0.003253615 | 0.000729123 | −0.002524492 |
| 56 | 0.005590292 | 0.000506243 | −0.005084050 |
| 61 | 0.004029631 | 0.000627639 | −0.003401992 |
| 62 | 0.004166453 | 0.000596618 | −0.003569834 |

There is a dimensional reason to distinguish this local budget from the whole retained energy. The shell has $S$ readings and the later span has dimension $S/2$. Its full annihilator therefore has dimension $S/2$, whereas the disjoint four-reading construction uses only $S/4$ independent functionals. It samples half of those directions; there is no corresponding assertion that it captures half the energy.

If a matrix $A$ spans the full annihilator, Proposition 9.4 gives the minimum retained energy as

$$(Av_S)^*(AW^{-1}A^*)^{-1}(Av_S).$$

This uses all invariant readings. The uniform comparison of the fragment excess with this minimum remains open. The failed local comparison does not settle it. The full increment also retains the aligned work and the remaining joined field change in (89).

**Problem 55** (A complete set of shell invariants)**.** At $S = 4$ use the later-family space $V = \operatorname{span}\{f_{1,5}, f_{1,7}\}$ on addresses $5, 6, 7, 8$ and the native shell weights. For the test inherited vector $v = (10, 7, 0, 0)^T$, evaluate the four-reading functional $L = (56, -80, 3, 21)$. Construct a second independent functional $J$ that annihilates $V$, using only the first two addresses. Form the joint matrix for rows $L, J$ and determine the minimum of $\|v + h\|_w^2$ over $h \in V$. Explain the information missed by $L$ alone.

In Figure 49, one invariant misses the field and another detects it. Problem 55 calculates the minimum energy from their joint work form.

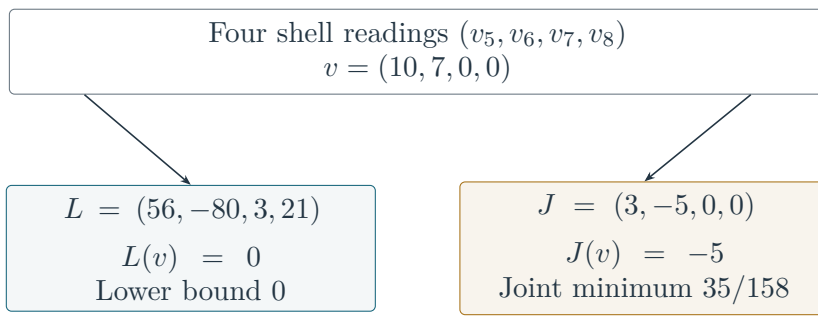



FIGURE 49. A missing reading can hide retained energy. At $S = 4$, the single four-reading functional gives zero on this test vector. A second independent invariant detects energy that the first misses. Their overlap is included in the joint matrix, giving minimum 35/158. This is a finite test vector, not the prescribed arithmetic source.

29.3. **Uniform increment estimate.** With $\ell_t = 1 + \lfloor \log_2 t \rfloor$, the remaining estimate is

$$\boxed{R_{4t}^{\mathrm{nat}} \le C\ell_t^3 \qquad (t \ge 1).} \tag{97}$$

Since $|W_S(M_S)| \le 516$, this estimate is equivalent, up to a changed absolute constant, to $B_{2t} - B_t \le C'\ell_t^3$. The estimate is open. Summing it along a doubling chain would give a fourth-power logarithmic energy bound; that energy bound alone would not control each increment.

Uniform control must account for the changing field. Fixed-field quadrature constants depend on its boundary data and curvature norms. The half-cell parity transfer addresses another field and observation; it does not bound the native remainder in (86).

The three forms of the remaining problem are now explicit. One may bound the complete increment, its finite head $H_t$, or the remainder $R_{4t}^{\mathrm{nat}}$ after the short cycles are screened. The tail and screening allowances prove the stated equivalences. The shell invariants constrain one part of that work; its correlation with the remaining terms still belongs to the complete energy change.

## 30. Completing an energy account

The ramp in Section 2 showed that one assigned reading can reverse the sign of a sampling defect. A second small example now brings the moving-field calculation to a close. This time the field changes, the observation loses an address, and infinitely many later readings remain. A short truncation again suggests the wrong sign. The complete account decides it.

Return to the two-clock field of Section 10,

$$p(u) = \{u\} - 2\{u/2\}.$$

At integer addresses its actual right-hand readings are $-1$ on odd integers and zero on even integers. Observe them with $w_n = 1/[n(n+1)]$. Double the field and move the cutoff from zero to one. The energy change is

$$\Delta E = E_1[2p] - E_0[p].$$

The reading at one has been removed. On every remaining odd address the squared reading rises from one to four.

30.1. **The removed reading and the common tail.** The old reading at one contributes $-1/2$ to the change. The field increment is $p$, so its mixed contribution on the common tail is $2E_1[p]$ and its new square is $E_1[p]$. Hence

$$\Delta E = -\frac{1}{2} + 2E_1[p] + E_1[p].$$

The two positive terms have different origins even though they contain the same tail sum. Dropping the inherited interaction would change the answer.

The complete series was evaluated earlier by integrating a geometric sum,

$$E_0[p] = \sum_{r\geq 0} \frac{1}{(2r+1)(2r+2)} = \log 2.$$

Thus $E_1[p] = \log 2 - 1/2$ and

$$\Delta E = 3\log 2 - 2.$$

This is a complete answer for the prescribed infinite observation. We can also determine its sign using only a finite calculation and the mass of the omitted tail.

30.2. **A finite calculation with its remainder.** Retain all changed readings through $N \geq 1$ and put

$$H_N = -\frac{1}{2} + 3 \sum_{\substack{1<n\leq N \\ n \text{ odd}}} \frac{1}{n(n+1)}.$$

At $N = 4$, only address three has been admitted after the removed reading. The result is $H_4 = -1/4$. Every omitted contribution is nonnegative and at most $3w_n$. The telescoping mass identity gives

$$H_N \leq \Delta E \leq H_N + \frac{3}{N+1}. \tag{98}$$

For four addresses the interval is $[-1/4, 7/20]$, which contains both signs. The negative finite head therefore cannot decide the sign of the complete change.

At twenty addresses, direct addition of the ten odd-address terms, including the removed reading, gives

$$H_{20} = \frac{489967}{77597520} > 0, \qquad H_{20} \leq \Delta E \leq H_{20} + \frac{1}{7}.$$

The whole interval is positive. The sign is now determined without discarding the remaining infinitely many readings. Increasing $N$ shrinks the allowance $3/(N+1)$ and can determine the energy change to any prescribed accuracy. The answer and its accuracy both refer to the same source and observation.

Figure 50 shows the enclosures narrowing around the complete answer. The first positive lower endpoint settles the sign while retaining the entire infinite tail.

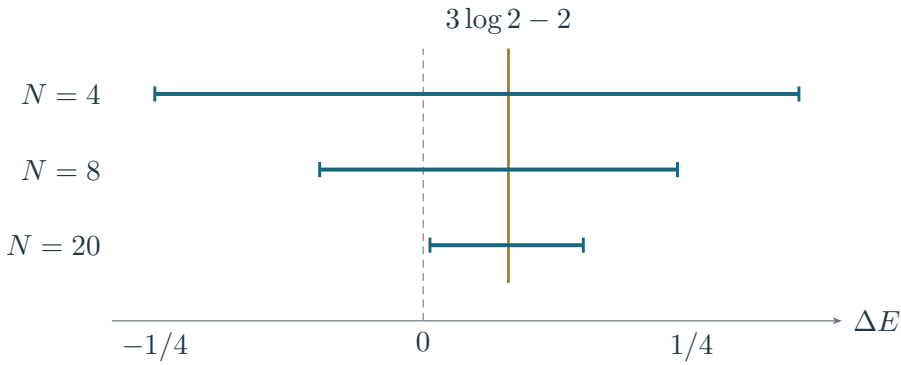



FIGURE 50. Finite heads with their complete tail allowances. The intervals at four and eight addresses cross zero. At twenty addresses even the lower endpoint is positive, so the finite calculation determines the sign of the full infinite change. The gold line gives its evaluated value for comparison.

30.3. **The growing arithmetic question.** This small field permits the entire calculation because its readings and tail envelope are fixed. In a growing family, that envelope must be estimated at each scale. The divisor application does this before choosing its quadratic cut-off. Its finite head $H_t$ differs from the complete increment $B_{2t} - B_t$ by a uniformly bounded tail allowance. A bound for that head is therefore equivalent to (97), with the parent shell and every mixed contribution still present.

The two-clock example shows the end of such an argument. Its finite head and tail allowance eventually decide the sign together. For the growing divisor source, the tail allowance is already uniform. The remaining task is to bound the finite head at every scale, with the parent shell and every mixed contribution retained. The reduction has given that task a finite form without discarding any part of the original observation.

## Unity and Plurality

This book began with the six fractions $1/7, 2/7, \ldots, 6/7$, read in bases two and three. Both collision tables were exact. Their different counts came from different ways of resolving the same fractions. Understanding either table was one task. Understanding the relation between them was another.

An integral and a finite observation can likewise give different energies for the same field. Each answer follows from its definition. The difference opens a further question about what the two descriptions reveal and what neither, on its own, explains. That question has mathematics of its own.

Clarity grows when we can place the descriptions in relation, see what each resolves and give the unresolved part a definite form. A more precise value within one description may leave that relation untouched. Knowing every entry of a table does not by itself explain how the table changes under refinement. Knowing a limit leaves open how the finite values approach it. Each calculation can be complete within its question while our understanding of the whole remains incomplete.

A fuller picture sometimes requires a larger mathematical object. The field gives values. The observation gives those values a place and a weight in the calculation. Their relation determines the energy we obtain. Once the observation is included, a change of grid, a moved boundary or a changed weight becomes something we can follow explicitly. What had appeared to be a disagreement between answers becomes a relation between specified operations.

The same enlargement is needed when components meet. Their separate energies leave the interaction unresolved. The mixed terms give that interaction a mathematical form. Cancellation then has a location in the calculation and a quantity to be estimated. The picture becomes more complete because it includes how the parts act together.

Unity gives us a reference for that completeness. A mass-one observation remains of mass one as it is subdivided. With inherited readings, subdivision preserves energy too. These are definite conservation statements, but they do not determine every reading on

the finer partition. Conservation and distribution ask different questions. Understanding their relation requires us to follow what the subdivision preserves and what its new distinctions make visible.

The error term belongs to this inquiry. A bound limits the size of a contribution that has not yet been fully resolved. Inside it may lie a signed interaction, a boundary correction or another term with a scale of its own. Recovering that structure enlarges the account. It can explain how a limiting law takes finite form, or why a contribution persists as the weight of each reading tends to zero.

An identity can expose such a relation before we know how to control it through every scale. The distinction is part of clarity too. Knowing the object of the next estimate is an advance in understanding; obtaining the estimate remains a separate mathematical achievement. A more complete picture includes a definite account of what is still unresolved.

To extend the mathematics here is to make more of these relations available to calculation. A correction becomes an object. An interaction becomes a form. A change of resolution becomes a map whose effect on the observation can be followed. Each extension lets us ask something that the separate descriptions left out. The growth of understanding lies in what these new relations allow us to explain.

The question of how a continuous whole finds expression in the finite therefore reaches beyond the number of readings we take. It concerns what those readings can carry, and what mathematics we need to understand their connection to the whole. Unity and plurality name the two sides of that inquiry.

Every distinction gives us something to see. The deeper work is to understand what holds through the distinctions. A whole becomes more intelligible as we recover the relations among its parts, including those that remain hidden when each part is read alone.

The boundary of a description can become the beginning of a larger mathematics. What we have not yet resolved gives us a place to look.

## 31. Sources of the methods

The opening's sampling comparison refers to Shannon's band-limited reconstruction theorem [25]. Its physical questions refer to the CODATA Planck scale [11] and the distinctions between limited resolution and discreteness discussed by Hossenfelder [7]. They motivate the relation between finite and continuous descriptions. The mathematical results in this book concern the specified fields and observations.

The opening digit invariant, cubic law and secondary term are results of [15, 17, 20]. The unit sampling defect is the later result of [21]. Their statements set the arithmetic problem; the opening's small finite counts are worked directly. The longer analytic proofs remain in those sources.

The preprint references also link to their Lean companions in nfield [19], at a fixed source revision. Partial coverage is marked in the bibliography. A companion is an entry point to the formalization. It may contain proof bodies or collect results proved in imported modules. An alias gives an existing declaration another name; its proof remains in the defining module. To inspect that proof, follow the fully qualified declaration to its source file.

The formal statement must be read with its definitions and assumptions. Checking its proof and checking that it expresses the paper's claim are distinct tasks. The companion links provide access to the cited formalizations; they do not constitute a formalization of this book.

The general operations come from established analysis and linear algebra. Their classical names identify the mathematics being used.

| Operation | Mathematical foundation |
|---|---|
| Boundary decomposition | Euler–Maclaurin summation with jumps and Bernoulli kernels [3, 4]. |
| Fourier reconstruction | Fourier series, finite Fourier inversion and Bernoulli expansions [26, 10]. |
| Field and observation change | Expansion and polarization of a squared norm; integration against atomic and continuous measures. |
| Weighted boundary work | Summation by parts, telescoping and finite differences [6, Chapter 2]. |
| Invariant readings | Weighted Cauchy–Schwarz and minimum-norm solutions under linear constraints [5, Sections 6.2 and 10.1.1]. |
| Measured recovery | Orthogonal projections, adjoints and spectral decomposition in an inner-product space [2, Chapters 6 and 7]. |
| Arithmetic registers and transforms | Congruence solving, gcd grouping and Möbius inversion [1, Chapters 2 and 5]. |

The point-value rule is the contribution of an atom to an integral. The positive midpoint term is the parallelogram identity. The localization rule compares two nonnegative tail energies. Each is derived here in the form needed for the calculations; its elementary basis remains visible.

For further study, Apostol [1] develops the divisor arithmetic behind the finite transforms. Montgomery and Vaughan [9] continue to Dirichlet series and analytic prime counting. Rademacher and Grosswald [23] treat the Dedekind sums that record the clock correlations. Iwaniec and Kowalski [8, Chapter 11] develop the complete exponential-sum estimates used in the analytic background to the carry application. These readings supply a wider setting for the methods; the specialized sampling estimates retain the sources identified below.

Unity in Section 2.5 is the usual constant function one. Its approximation uses weighted squared error, and its mass partitions use additivity of the observation. The arithmetic setting and the

distinction between resolving a reading and admitting a new clock come from the refinement work [22].

The contribution of the general presentation is to put these operations on the same specified field and observation. A boundary reading, a combined reset, a moved sampling grid and a changed activation cutoff can then be followed through one calculation. The clock formula makes that organization explicit through the collision amplitude, inherited reading and slope. The moving-window and pulse examples show how to turn the identities into complete answers for growing families.

Several parts of the earlier arithmetic development are treated only briefly here. The general passage from weighted digit rules to signed boundaries is developed in *Carry Boundaries and Bernoulli Spectra in Long Division* [14]. The conserved period mass and its profinite state are developed in *Conservation and Profinite Dynamics in Digit-Collision Energy* [16]. *Finite Generation of Collision Capacity* [18] studies selection of that mass and recovery of its primitive kernel from the complete capacity sequence, with the class restrictions and sensitivity estimates that recovery requires.

The alignment count and deficit-ratio formulas in Section 13 come from *The Alignment Deficit Lattice and Musical Tuning* [13]. The prime-exponent description of just intonation is classical; the cited work derives its particular deficit coordinate from repeating digits. The binary-field realization here expresses the same finite count as an energy difference, retaining the omitted address and the renormalization of the observation.

Sections 14–18 collect tools developed in the square-root sampling investigation. The signed register, finite-column estimate and interior norm construction retain that investigation's specific arithmetic data. Primitive-shell grouping and the logarithmic transform express classical divisor operations in a reusable form; their application to the clock remainder supplies the finite prime-counting identity. The derivations are included here.

The specialized results have a more particular provenance. The unshifted carry limit comes from *The Unit Sampling Defect in Digit-Collision Energy* [21]. The half-cell source reduction, parity transfer and quarter-limit are developed in Section 26, using analytic estimates from that work. They first appeared in the early digital text.

The native screening constant 129, cycle-alignment credit and parent-shell constructions were developed in the divisor-field investigation. Sections 27–29 derive them from the prescribed family profiles and include their proofs. The quadratic cutoff is an explicit application of the general localization rule. The four-reading construction specializes constrained energy to the later divisor families. Its failed sufficiency comparison at the four stated cutoffs is an exact result of the present investigation. These are specific formulas, constructions and estimates, with their derivations and dependencies identified. Collision calculus names this coordinated method of boundary and energy accounting.

The finite examples in Part 4 develop the linear algebra of measured lifts and transported densities directly. Cyclic Fourier inversion, finite translation differences and adjoint maps supply the remaining general tools. Their proofs here specify the source, observation and inverse costs needed for use in another problem.

## Hints and solutions

The answers follow the problem numbers, grouped by part. Each answer heading links back to its question. Where a hint is provided, try it before reading the full solution.

### Part I. Collision energy and boundary readings

Problem 1. *Solution.* Only the first squared reading changes. The sampled energy is $c^2/3+2/27$, and the discrepancy is $7/27-c^2/3$. It vanishes exactly when $c=\pm\sqrt{7}/3$. No integral needs to be recomputed.

Problem 2. *Solution.* Equation (1) gives $D_q[f_q]=q^{2\alpha-1}(1-2/(3q))$. The three cases are $\alpha<1/2$, $\alpha=1/2$ and $\alpha>1/2$, respectively.

Problem 3. *Hint.* Use the observation masses both in the mean and in the squared error. Subtract unity before squaring the error.

*Solution.* The weighted mean is $1+(a+2b)/3$, so it equals one when $b=-a/2$. In that case

$$\frac{1}{3}(1+a)^2+\frac{2}{3}(1-a/2)^2=1+\frac{a^2}{2},\qquad \frac{1}{3}a^2+\frac{2}{3}(-a/2)^2=\frac{a^2}{2}.$$

The weighted detail has mean zero, so its mixed term with unity vanishes. For $a=2$, the readings are $3,0$, the energy is three and the error energy is two. With $b=-a$, the mean is $1-a/3$, the energy is $1-2a/3+a^2$, and the error energy is $a^2$. Opposite changes need not have zero weighted mean. The total mass stays one in both cases. Copying the parent reading one would also preserve its energy; changed child readings require the new square.

Problem 4. *Hint.* Determine whether each grid samples zero.

*Solution.* Check whether the exceptional point is sampled. The continuous energy is zero. The unshifted grid has sampled energy $4/4=1$ and defect $-1$. The half-cell grid misses zero, so both its sampled energy and defect vanish. The assigned value at an atom distinguishes the observations.

Problem 5. *Hint.* Expand the new squared readings and keep the change of observation separate.

*Solution.* Keep the masses separate from the field values. The old energy is one; the new field is $(1, 0, 1)$ and its energy is $2/3$. The observation change is $3 - 1 = 2$, the inherited interaction is $-10/3$, and the new square is one. Their sum is $-1/3$, agreeing with the direct subtraction.

Problem 6. *Hint.* Form the squared readings before taking their average. At a boundary, square each trace separately.

*Solution.* At 0 the traces of $f$ are $3, 1$; at $1/2$ they are $1, 3$. The actual reading is two at both points. Thus the corresponding traces of $h$ are $9, 1$ and $1, 9$, and its actual reading is four. Consequently $a_0(h) = a_{1/2}(h) = 1$ and $[h]_0 = -8$, $[h]_{1/2} = 8$. All slope traces vanish. The continuous energy is five. On the three-point grid the readings are $(2, 1, 3)$; on the four-point grid they are $(2, 1, 2, 3)$. Hence

| $q$ | $Q_q(f^2)$ | $D_q[f]$ | $P_q[f]$ |
|---|---|---|---|
| 3 | $14/3$ | $1/3$ | $1/3$ |
| 4 | $9/2$ | $1/2$ | $1/2$ |

Only zero is a sampled boundary for $q = 3$; both are sampled for $q = 4$. In both cases $Q_q f = 2$, so $(Q_q f)^2 = 4$ differs from the observed energy. Here the remaining signed defect is zero.

Problem 7. *Hint.* Apply boundary decomposition to the squared field.

*Solution.* Apply the boundary formula to $h = x^2$, rather than to $f$. The point correction is $1/2$, the value jump is $-1$, the derivative jump is $-2$, and the regular curvature is the constant two. The value-jump term vanishes on the unshifted grid. The regular term has zero Bernoulli mean. Thus

$$D_q[f] = \frac{1}{2q} - \frac{1}{6q^2}.$$

Directly, subtract the sampled sum from the integral:

$$D_q[f] = \frac{1}{3} - \frac{1}{q^3} \sum_{a=0}^{q-1} a^2 = \frac{1}{2q} - \frac{1}{6q^2}.$$

Problem 8. *Hint.* Use $\sin^2 t = (1 - \cos 2t)/2$ and the finite Fourier-mode average.

*Solution.* The integral is $1/2$. Every reading at $q = 2m$ is zero, so the defect is $1/2$. At $q = 2m + 1$, the mode of frequency $2m$ has zero grid average; the sampled energy is $1/2$ and the defect is zero. The same argument gives zero defect for every fixed-field grid $q > 2m$. There are no jumps, exceptional readings or derivative jumps. The regular-curvature integral in Theorem 5.1 therefore supplies the whole defect on the resonant grid. Along $q = 2m$ the frequency grows with the grid, so the fixed-field conclusion does not apply to that sequence.

Problem 9. *Solution.* Point-value replacement and the single-clock formula give

$$D_q[g] = D_q[f] - \frac{1}{q} = -\frac{2}{3q^2}.$$

The two energy traces at zero equal one, as does the new sampled energy. The point correction has disappeared. The remaining derivative term is negative, so the grid records more energy than the integral.

Problem 10. *Hint.* Write $F$ on the two open half-intervals and combine the coefficients at a shared reset before squaring.

*Solution.* The possible resets are $0, 1/2$. Their combined coefficients are $C_0 = 1 - 1 = 0$ and $C_{1/2} = -1$. The field is

$$F(x) = \begin{cases} -2x, & 0 < x < 1/2, \\ 2 - 2x, & 1/2 < x < 1, \end{cases} \qquad F(0) = F(1/2) = 0.$$

It has continuous energy $1/3$. Its four readings are $(0, -1/2, 0, 1/2)$, so the sampled energy is $1/8$ and the defect is $5/24$. The first component has the same four readings and continuous energy; the second has continuous energy $1/3$ and all four readings zero. Thus

$$D_4[f] + D_4[g] = \frac{13}{24}, \qquad 2\,\mathrm{Re}\,K_4(f, g) = -\frac{1}{3}.$$

At zero the resetting coefficients cancel, so that point has no combined jump or point-energy correction. Squaring the separate coefficients would lose this cancellation.

Problem 11. *Solution.* Each interval has length $1/2$ and contains two of the five sample points. Each individual defect is $1/2-2/5=1/10$. The intersection has length $1/4$ and contains one sample, so its defect is $1/20$. Since $f^2=\mathbf{1}_I+\mathbf{1}_J+2\mathbf{1}_{I\cap J}$,

$$D_5[f]=\frac{1}{10}+\frac{1}{10}+2\frac{1}{20}=\frac{3}{10}.$$

Directly, the field has readings $0,0,1,2,1$, giving sampled energy $6/5$ and continuous energy $3/2$. Adding only the two individual defects would miss $1/10$ of the discrepancy.

Problem 12. *Hint.* Inspect the square of the combined field before replacing either reading.

*Solution.* Square the combined field first. Its square is one everywhere, so $D_q[f]=0$. Both changed points are sampled when $q$ is even. Removing their two unit squares reduces the sampled energy by $2/q$ and leaves the integral unchanged. Hence $D_q[g]=2/q$.

Problem 13. *Hint.* Count the grid points in the half-open pulse separately for each residue class of $q$.

*Solution.* Count samples before scaling. The number is $\lceil q/3\rceil$. For $q=3m$ the defect is zero; for $q=3m+1$ it is $-2/(3q)$. Both fixed-field defects tend to zero. Multiplication of the field by $\sqrt{q}$ multiplies its defect by $q$, giving the distinct values zero and $-2/3$. The growing family has no common limit along these sequences.

## Part II. Field and observation changes

Problem 14. *Solution.* Extend $f$ through $b+1$. The observation change contributes

$$-v_{a+1}f(a+1)^2+v_{b+1}f(b+1)^2.$$

On the new window add

$$\sum_{a+1<n\le b+1} v_n(2f(n)g(n)+g(n)^2).$$

These are exactly the three terms of (3). Changing masses at retained addresses would add their differences times $f(n)^2$ to the first term.

Problem 15. *Hint.* Apply summation by parts to the product $w_n v(n)$, not to $v(n)$ alone.

*Solution.* The primitive is $(1, 0, 1, 0, 1)$ and the direct sum is

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20} + \frac{1}{30} = \frac{2}{5}.$$

Summation by parts gives

$$T(5)w_5 + \sum_{n=1}^{4} T(n)(w_n - w_{n+1}) = \frac{1}{30} + \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{12} - \frac{1}{20}\right) = \frac{2}{5}.$$

The partner variation vanishes, but the weight variation does not. Dropping the endpoint gives $11/30$; dropping the interior terms gives $1/30$. Neither is the observed sum.

Problem 16. *Hint.* Track the offset of the three distinct arguments after dilation, as well as their multiplicity.

*Solution.* For $r = 2$, the half-cell grid visits $1/6, 1/2, 5/6$, each twice. Their readings are $-2/3, 0, 2/3$, so the sampled energy is $8/27$ and the defect is $1/27$. For $r = 4$, it visits $0, 1/3, 2/3$, each twice. Their readings are $0, -1/3, 1/3$, so the sampled energy is $2/27$ and the defect is $7/27$. Dilation preserves the continuous energy $1/3$. At offset zero both dilations visit $0, 1/3, 2/3$ twice and give defect $7/27$. The reduced grid has three points in both cases, but its offset is $\{r\theta/2\}$: it is $1/2$ for $r = 2$ and zero for $r = 4$ when $\theta = 1/2$.

Problem 17. *Solution.* Every entry of the defect matrix is $d_q = 1/q - 2/(3q^2)$, so

$$H = d_q \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad U^* H U = \begin{pmatrix} 2d_q & 0 \\ 0 & 0 \end{pmatrix}.$$

The symmetric coordinate represents $\sqrt{2} f_1$; the antisymmetric coordinate represents the zero field. The sampled point matrix is $q^{-1} \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The same basis change diagonalizes it. Its trace is preserved, but its two diagonal energies are unequal. This elementary case distinguishes a unitary invariant from an equal-distribution assertion.

Problem 18. *Solution.* When $q = m$, every sampled point is a reset. The sample energy is zero and $D_m[f_m] = 1/3$. When $q = m+1$, the two integers are coprime. Dilation reduces the samples to those of the unit clock, giving

$$D_{m+1}[f_m] = \frac{1}{m+1} - \frac{2}{3(m+1)^2} \longrightarrow 0.$$

The number of samples alone does not identify the limiting regime. The common divisors with the growing frequency determine which readings the grid repeats.

Problem 19. *Solution.* Here $b = (-2, 1)^T$ and

$$G = \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix}, \qquad G^{-1} = \frac{1}{20}\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}.$$

The joint bound is $b^* G^{-1} b = 7/10$. The kernel of $A$ consists exactly of affine sequences, so this is the minimum. It is attained at $z = (-4, 7, -2, -1)/10$, obtained by adding $h(i) = (i - 5)/10$. The two separate bounds sum to $4/6 + 1/6 = 5/6$, which exceeds the attained energy. Their common addresses require the mixed entries of $G$.

Problem 20. *Hint.* Find the weighted mean, then test two differences that annihilate constants.

*Solution.* A constant adjustment removes the weighted mean, which is $1/4$. The minimizing addition is $-1/4$, and the resulting field has energy

$$(3/4)^2 + 2(-1/4)^2 + (-1/4)^2 = 3/4.$$

Use readings $v_1 - v_2$ and $v_2 - v_3$. With $W = \operatorname{diag}(1, 2, 1)$,

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \; G = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$

Since $Af = (1, 0)^T$ and $\ker A$ is the constant space, the joint bound is the attained minimum $3/4$.

Problem 21. *Hint.* Minimize the energy over a constant addition and compare it with the proposed reading.

*Solution.* The single reading gives the valid lower bound zero. A real constant addition $c$ has energy $c^2 + (1+c)^2 + c^2 = 3(c+1/3)^2 + 2/3$. An imaginary part adds three times its square, so the minimum over complex additions is also $2/3$. The first and third readings agree, but their common difference from the middle reading survives every

constant addition. That is the information the chosen functional misses.

Problem 22. *Hint.* A constant reading has native tail mass $1/(S+1)$.

*Solution.* Use $\sum_{n>S} w_n = 1/(S+1)$. The old energy is $1/2$ and the new energy is $(1/2)^2/4 = 1/16$, so the change is $-7/16$. Its three contributions are $-1/4$, $-1/4$ and $1/16$. They come respectively from the removed addresses two and three, the constant mixed term on the remaining tail, and the new square.

Problem 23. *Hint.* Bound the omitted new and old energies separately before subtracting.

*Solution.* The omitted change lies in $[-4/(N+1), 9/(N+1)]$. Both endpoints have absolute value at most $1/10$ when $N+1 \geq 90$; the least admissible cutoff from these envelopes is $N = 89$. A head of $-1/10$ then gives a nonpositive complete change, with interval $[-13/90, 0]$. Strict negativity does not follow. With envelopes $2S, 3S$ the condition is $N+1 \geq 90S^2$, so $N = 90S^2 - 1$ suffices and is at least $2S$.

Problem 24. *Solution.* In the notation of Section 11, the exact expression is $U_N + 2N^{\alpha}V_N + N^{2\alpha}U_N$. The proof of Theorem 11.1 gives $U_N = 1/(2N) + O(N^{-2})$ and $V_N = O(N^{-2})$. For $\alpha < 1/2$ all terms tend to zero. At $\alpha = 1/2$ the limit is $1/2$. For $\alpha > 1/2$ the last term is asymptotic to $\frac{1}{2}N^{2\alpha-1}$ and dominates the other two, so the energy diverges. The threshold comes from the observation mass as well as the field's amplitude.

Problem 25. *Hint.* Inspect partial sums of both the carrier and its squared fluctuation over one period.

*Solution.* Inspect the range of each period's partial sums. For $u$ the range has width one. For $u^2 - 2/3$, with period $(1/3, 1/3, -2/3)$, it has width $2/3$. Thus $P = 1$, $R = 2/3$, $c = 2/3$ are admissible. All three products $h_i$ equal one, so their variations vanish and $\|h_i\|_{\mathrm{BV},*} = 1$. The theorem gives

$$\left| E_N - \frac{2}{3} - \frac{1}{N} \right| \leq \frac{2}{3N} + \frac{2}{N^{3/2}}.$$

The limit is $2/3$. At $N = 4$ the readings are $(-1, 1, 3, -1)$, with common weight $1/16$, giving $E_4 = 3/4$.

Problem 26. *Hint.* Identify whether the changed object is a point reading, weighted partner, tail or growing field.

*Solution.* Match the operation to the changed data. An isolated reading uses point-value replacement and its sample mass. Periodic work uses summation by parts with an anchored carrier primitive and the variation of its weighted partner. A finite head uses localization with both tail energies bounded. For a growing family, track every boundary amplitude, curvature norm and other field-dependent constant as a function of scale. Uniform control of those constants is the additional requirement.

Problem 27. *Hint.* Retain the mixed term before substituting the two amplitudes.

*Solution.* Retain the mixed energy and put $M = q(a+b)^2/2$. The continuous energy is $M - ab$. The sampled energies are $M - 2ab$ for $\theta < 1/2$ and $M$ for $\theta \geq 1/2$, so the defects are $ab, -ab$. They vanish for every offset exactly when $ab = 0$. The difference gives defects $-1, 1$; the sum gives $1, -1$ and continuous energy $2q - 1$.

Problem 28. *Hint.* Telescope the weights for the two squares. Pair successive terms of the alternating tail before estimating the mixed term.

*Solution.* Let $T_N = \sum_{n>N} w_n(-1)^n$. Expansion of the whole square gives

$$\mathcal{E}_N = \frac{a^2 + N}{N+1} + 2a\sqrt{N}\, T_N.$$

The weights decrease to zero. Pairing adjacent terms, starting with the first term or with the second, places the alternating sum between zero and its first signed term. Hence

$$|T_N| \leq w_{N+1} = \frac{1}{(N+1)(N+2)},$$

$$\left| \mathcal{E}_N - 1 - \frac{a^2 - 1}{N+1} \right| \leq \frac{2|a|\sqrt{N}}{(N+1)(N+2)}.$$

All series converge absolutely for each fixed $N$, and the displayed allowance proves $\mathcal{E}_N \to 1$. For $a = 1$, the envelope gives

$$\sum_{n>M} w_n |p_N(n)|^2 \leq \frac{(1+\sqrt{N})^2}{M+1} \leq \frac{4N}{M+1}.$$

The integer choice $M_N = 4N^2$ therefore suffices. A cutoff proportional to $N$ leaves only an $O(1)$ allowance from this envelope. The desired $1/N$ accuracy requires a quadratic cutoff by this particular method.

## Part III. Finite arithmetic tools

Problem 29. *Solution.* The deficit ratio is $(33t - 1)/(22t - 1)$. Its absolute excess over $3/2$ is $1/[2(22t - 1)]$ and its relative excess is $1/[3(22t-1)]$. At $t = 1$, these are $32/21$, $1/42$ and $1/63$, respectively. The numerator $v - u = 1$ is positive, so the finite ratio approaches the fifth from above along every supported scaling sequence.

Problem 30. *Solution.* Removing address zero with its old mass reduces energy by $1/(pm)$, leaving $(2m - 1)/(pm)$. Normalization multiplies all remaining masses by $pm/(pm - 1)$ and increases that energy by $(2m - 1)/[pm(pm - 1)]$. The net loss is

$$\frac{1}{pm} - \frac{2m - 1}{pm(pm - 1)} = \frac{p - 2}{p(pm - 1)}.$$

The omitted reading and changed masses are both part of the observation change. Omitting the second step gives a different deficit.

Problem 31. *Solution.* Both components are read only at $r = 25$. Their combined reading is $2 - 3 = -1$, so the energy is one. The ordered terms are $4, 9, -6, -6$, whose sum is one. Discarding the two mixed signs would give 25.

Problem 32. *Hint.* Solve the new pair of congruences and then test the observation interval.

*Solution.* The new residue is 39 modulo 56, since $39 \equiv 4 \pmod 7$ and $39 \equiv -1 \pmod 8$. Both systems are compatible, but the original kernel on $24 < r < 27$ is one and the new kernel is zero. Changing an orientation can change the observed intersection.

Problem 33. *Hint.* List the common rows first, and then count their observation masses.

*Solution.* The common class is 10 modulo 12, so both rows ten and twenty-two are observed. Their total mass is three. The first component occupies nine rows and the second six; the extra mass at

row ten gives

$$K_v = \begin{pmatrix} 10 & 3 \\ 3 & 7 \end{pmatrix}.$$

The two energies are

$$(2,-1)K_v(2,-1)^T = 40 - 12 + 7 = 35,$$
$$(2,1)K_v(2,1)^T = 40 + 12 + 7 = 59.$$

The sign change adds twenty-four through the mixed terms. The observation interval is longer than the common period twelve, so the intersection kernel must sum both common rows and their actual masses.

Problem 34. *Hint.* Balance the two powers in the allowance.

*Solution.* Write $J = p^\theta$. The costs have exponents $\theta$ and $1 - \theta/3$, which agree at $\theta = 3/4$. Taking an adjacent integer width gives allowance $O(p^{3/4})$ and normalized rate $O(p^{-1/4})$. This conclusion uses the two assumed costs; it does not estimate a new smooth family.

Problem 35. *Hint.* The arguments may differ by an integer. Use $|e(t) - 1| = 2|\sin(\pi t)|$ for the omitted factor.

*Solution.* Both inverses are three in their respective moduli. The left argument is $9/7$ and the full right argument is $-9/5 + 3/35 = -12/7$; their difference is three. Thus the two phases agree. Omitting the correction leaves absolute error

$$|e(9/7) - e(-9/5)| = |e(3/35) - 1| = 2\sin(3\pi/35).$$

In general the other phase has unit magnitude, so the error is

$$2|\sin(\pi A/(rc))| \le \frac{2\pi|A|}{rc}.$$

It is also at most two. This estimate concerns the phase correction alone; multiplying or summing it requires retaining the accompanying amplitudes.

Problem 36. *Solution.* The coefficients are $1, 2, 4, 4$. For $m > 1$ there are two orientations of each coprime pair $(k, m)$, $1 \le k < m$. The multiplicities at cutoff four are $4, 2, 1, 1$. Thus $\mathcal{A}(4) = 4+4+4+4 = 16$, as direct counting gives.

Problem 37. *Hint.* Compute the four primitive shells and retain the prime-power contribution at four.

*Solution.* The nonzero direct terms are $4\log 2 + \log 3$. The shell formula gives $\Psi(4) + 2\Psi(2)$ because the later arguments are below two. Since $\Psi(4) = 2\log 2 + \log 3$, the same value follows. Both calculations retain the repeated contribution of the prime two at the prime power four.

Problem 38. *Hint.* Reduce each nonzero pair by its common divisor. Only three primitive pairs remain.

*Solution.* The nonzero primitive pairs are $(1,1), (1,2), (2,1)$. Thus $P(1) = 2$, $P(2) = -2$ and $P(m) = 0$ for $m > 2$. Common dilation preserves each of the three conditions defining $R$. The prefix is

$$\mathcal{A}(M) = 2M - 2\lfloor M/2 \rfloor = 2\lceil M/2 \rceil,$$

so its first six values are $2, 2, 4, 4, 6, 6$. The nonzero logarithmic terms at $N = 6$ give

$$\mathcal{K}(6) = 4\log 2 + 2\log 3 + 2\log 5 - 2\log 6 = 2\log 10.$$

The shell formula gives the same value as $2\Psi(6) - 2\Psi(3)$, since $\Psi(6) - \Psi(3) = \log 2 + \log 5$. At every even entrance $M$, the added diagonal contribution two is canceled by the two new off-diagonal readings $-1, -1$.

Problem 39. *Hint.* Mask the reset record before applying its inverse.

*Solution.* The retained record is $(2, 0, 1, 0)^T$, so the source is $(1, 0, 1, 0)^T$. Its continuous work is $8/9$. The complement $(0, -1, 0, 1)^T$ has work $1/3$, leaving mixed contribution $1 - 8/9 - 1/3 = -2/9$. In this particular selection, deleting the other two coefficients happens to give the same source, because their sum at the common reset is zero. The selection in the worked example shows why that coincidence is not a general rule.

Problem 40. *Solution.* Continuous orthogonality gives work $9(1/3) + 1 = 4$. Every sampled reading vanishes since $g_2 = -3g_1$ on this grid. The defect is four. In the sampled quadratic form, the two diagonal contributions sum to $4/3$ and the mixed term is $-4/3$. Dropping that term would give a different sampled energy.

Problem 41. *Hint.* Normalize the later term by the proposed allowance at $n$.

*Solution.* Solving for the value at $n$ divides by $c$. The later term has argument $an/b$, so its absolute allowance relative to $Cn^\alpha$ is $|d/c|(a/b)^\alpha$. For the specified shells this is $(5/4)^{1-\alpha}$, greater than one for $\alpha < 1$. At $\alpha = 1/2$ it equals $\sqrt{5}/2$. This term prevents contraction of that absolute induction; it does not determine the size of the signed recurrence.

Problem 42. *Hint.* Solve the identity at $x = 2n$ and iterate until the argument of $E$ is below one.

*Solution.* The recurrence is $E(n) = F(2n) + \frac{1}{4}E(n/2)$. With $J = \lfloor \log_2 n \rfloor$, its terminal value vanishes and

$$E(n) = \sum_{j=0}^{J} 4^{-j} F(2n/2^j), \qquad |E(n)| \le \frac{C\sqrt{2n}}{1 - 1/(4\sqrt{2})}.$$

Indeed, the termwise allowance has ratio $1/(4\sqrt{2}) < 1$. The arithmetic example in Problem 41 instead has ratio $\sqrt{5}/2 > 1$ at this scale. Here the specified coefficients permit the absolute induction; those arithmetic coefficients do not.

## Part IV. Measured geometry and refinement

Problem 43. *Hint.* Use $U = (1, 1)$ with the actual Gram, or substitute $c_2 = 1 - c_1$ directly in the quadratic form.

*Solution.* One has $G^{-1}U^T = (2/5, 1/5)^T$ and $UG^{-1}U^T = 3/5$. The lift is $(2/3, 1/3)^T$, with readings $(2/3, 1/3, 1)^T$ and work $5/3$. Its product with $G$ is $(5/3, 5/3)^T$, perpendicular to $(1, -1)^T$ in the ordinary dot product. This is the required $G$-orthogonality. Since $c = Az$, the detector is $UAz = z_1 + 2z_2$.

Problem 44. *Solution.* The map is $v = u/(2\theta)$ on the left and $v = 1/2 + (u - \theta)/(2(1 - \theta))$ on the right. Its inverse densities are $2\theta$ and $2(1 - \theta)$. The transported energy is therefore $\theta|a|^2 + (1 - \theta)|b|^2$, as before. Unit density would give $(|a|^2 + |b|^2)/2$. Subtracting the actual energy gives error $(1/2 - \theta)(|a|^2 - |b|^2)$, which can have either sign.

Problem 45. *Hint.* The alternating vector is an eigenvector of $D$ with eigenvalue two.

*Solution.* The mean-zero solution is $(1/2, -1/2, 1/2, -1/2)^T$. Adding mean three gives $(7/2, 5/2, 7/2, 5/2)^T$. Their first readings are $1/2$ and $7/2$, and their counting energies are one and thirty-seven. The curvature record agrees, but the missing constant contributes energy thirty-six.

Problem 46. *Solution.* The readings are $(0, 2, 0, 2)$, with mean square two. The source work is $1+4 = 5$. From the first two readings $y_0, y_1$, recover $a = (y_0 + y_1)/2 = 1$ and $b = (y_0 - y_1)/(2\delta) = -2$. At $\delta = 1/L$, the inverse factor for this coefficient grows as $L$ and the lower visibility constant is $L^{-2}$ for $L \geq 1$. The observation remains a four-event history; $L$ here indexes the changing detector gain, not the number of events.

Problem 47. *Solution.* The two transported readings are $O_1' R(a, b) = (a + b)/2$ and $O_2' R(a, b) = 0$. Both annihilate the old null direction $(1, -1)$ and factor through $O$ by multiplication by $1/2$ and zero respectively. At $(2, 1)$ the refinement error is $(1, 0, -1, 0)$, with mean-square work $1/2$.

Problem 48. *Solution.* The complete readings are zero, zero and $2ab$. At $z = 0$, removing the first member changes these to $-1$, zero and $2ab$, because the omitted readings are one, zero and zero. For $a = b = 1$, the kernel is $-1$ on $[-2, -1)$ and $1$ on $[-1, 0)$. Its mass is zero and its first moment is $-\int_{-2}^{-1} z\,dz + \int_{-1}^{0} z\,dz = 1$.

Problem 49. *Solution.* The signed reading becomes $-\delta$ and its square is $\delta^2$. The diagonal is $6 + 4\delta + \delta^2$ and the mixed term is $-6 - 4\delta$. Here $\|e\|_1 = \|e\|_2 = |\delta|$. The linear and scalar-square error bounds are equalities. The diagonal change obeys $|4\delta + \delta^2| \leq (2\sqrt{6} + |\delta|)|\delta|$. For a fixed slope and interval, choose $a = -bh/2$ to remove the mean-height square. The minimum local work is $b^2h^3/12$.

## Part V. Arithmetic energy at growing scales

Problem 50. *Hint.* The equal-digit positions are $0, 6, 12, 18, 24$, and the address multiplier is $-4$ modulo 25.

*Solution.* The direct calculations give

| $A$ | 0 | 1 | 5 | 12 |
|---|---|---|---|---|
| $a$ | 0 | 21 | 5 | 2 |
| $C_5(a)$ | 0 | 1 | 1 | 2 |
| $F_5(a)$ | 0 | $-16/5$ | 0 | $8/5$ |
| $B_4(A/25)$ | 0 | $-16/5$ | 0 | $8/5$ |

For example, at $a = 21$ the five floor differences are $(0, 0, 0, 0, 1)$; at $a = 2$ they are $(0, 0, 1, 0, 1)$. The multiplier $-4$ is invertible modulo twenty-five, so every address appears exactly once. Lemma 25.1 identifies each reading after this permutation, and summing their squares preserves the whole energy. The equality is a relabeling of the table.

Problem 51. *Hint.* Only $k = 3$ has a nonzero remainder when four is divided by $k = 2, 3, 4$.

*Solution.* Only $1/2$ is a sampled reset. Its resetting frequencies are two and four, so the point contribution is $2^2/25 = 4/25$. For $k = 3$, the incomplete block has one clock and $M_3 = 1$. The readings $T_{3,1}(1), T_{3,1}(2)$ are $-1/3, 1/3$, while the shifted phase readings are $1/3, -1/3$. Hence $\mathcal{A}_5^{\mathrm{half}} = -2/9$. Since $H_2 = 3/2$ and $H_4 = 25/12$,

$$D_{25,1/2}[B_4] = \frac{4}{25} - \frac{8}{225} - \frac{1}{1125} = \frac{139}{1125}.$$

The difference from the point contribution is $-41/1125$. The shifted boundary is visible, but its contribution alone does not give the finite energy defect.

Problem 52. *Hint.* A primitive difference gives the interval sum. For the weighted sum, retain the actual weight at each of the eight addresses.

*Solution.* The primitive values are

$$(0, -1, 1, 2, 1, -1, 0, 0, 0).$$

Its range has width three; the interval $2 \le n \le 3$ attains that value. The unweighted period sum is zero, whereas

$$\sum_{n=1}^{8} w_n H_1(n) = -\frac{1}{2} + \frac{1}{3} + \frac{1}{12} - \frac{1}{20} - \frac{1}{15} + \frac{1}{42} = -\frac{37}{210}.$$

A constant partner therefore need not have zero weighted interaction with a completed cycle. Summation by parts must retain variation of the observation weights, as well as any variation of the partner.

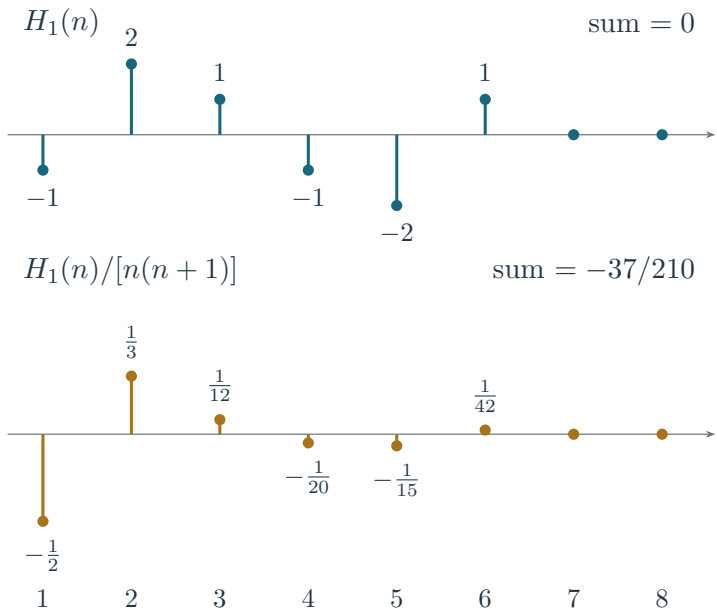



FIGURE 51. The same completed period under two observations. The eight readings sum to zero. Multiplying each by its native mass leaves $-37/210$. Vertical scales differ so that the smaller weighted terms remain readable; each stem is labeled by its value. A zero unweighted mean does not cancel the variation in the observation weights.

Problem 53. *Hint.* There are two summations. The odd positive integers through $M$ sum to $\lceil M/2 \rceil^2$.

*Solution.* Keeping the factor two in the mixed energy gives

$$|W_S(M)| \le 2(2S)\frac{129}{S^2}\lceil M/2 \rceil^2 = \frac{516}{S}\lceil M/2 \rceil^2.$$

At $S = 256$, the respective allowances are 129 and 516. For fixed $0 < \alpha < 1$, the condition $M \le S/8$ holds at all sufficiently large scales. The allowance has order $S^{2\alpha-1}$, so it is bounded for $\alpha \le 1/2$. Including every completed cycle means $M = \lfloor S/8 \rfloor$ and gives an $O(S)$ allowance. The count of changed families and the sum of the old cycle lengths consume the pairwise saving. This estimates the absolute sum; it is not a lower bound on the signed work.

Problem 54. *Hint.* The old field is $-1$ on odd addresses and zero on even addresses. The odd native masses sum to $\log 2$.

*Solution.* The new field is zero on odd addresses and one on even addresses. Only address three contributes to the removed shell. The three terms in the moving-field identity are

$$-\frac{1}{12}, \qquad -2\left(\log 2 - \frac{1}{2} - \frac{1}{12}\right), \qquad \frac{1}{5}.$$

Their sum is $77/60 - 2\log 2$. Directly, $E_2[p] = \log 2 - 1/2$ and $E_4[q] = 47/60 - \log 2$, which give the same difference. The identity $\sum_{n \text{ odd}} w_n = \log 2$ follows from the alternating harmonic series, or the two-clock integral in Section 10.

For the finite calculation, the difference of squares on $n > 4$ is $(-1)^n$. Thus

$$H(10) = -\frac{1}{12} - \frac{1}{30} + \frac{1}{42} - \frac{1}{56} + \frac{1}{72} - \frac{1}{90} + \frac{1}{110} = -\frac{137}{1386}.$$

Both field envelopes are one. Localization gives the complete enclosure

$$E_4[q] - E_2[p] \in \left[H(10) - \frac{1}{11},\ H(10) + \frac{1}{11}\right] = \left[-\frac{263}{1386}, -\frac{1}{126}\right].$$

Its upper endpoint is negative. This establishes the sign using a finite calculation and a proved tail allowance, without approximating $\log 2$ or omitting the parent shell.

Problem 55. *Hint.* Use the later-family formula to find its first two readings. Two independent annihilators suffice because the later-family space has dimension two.

*Solution.* The later-family matrix and a second annihilating row are

$$U = \begin{pmatrix} -3/4 & 0 \\ -9/20 & 0 \\ -1/10 & -1/4 \\ 3/10 & 1/28 \end{pmatrix}, \qquad J = (3, -5, 0, 0).$$

Both $LU$ and $JU$ vanish. The rows are independent, and $U$ has rank two. Thus, for $A$ with rows $L, J$, one has $\ker A = V$. The separate retained readings are $L(v) = 0$, $J(v) = -5$. With $W = \operatorname{diag}(1/30, 1/42, 1/56, 1/72)$, the joint matrix is

$$G = AW^{-1}A^T = \begin{pmatrix} 395136 & 21840 \\ 21840 & 1320 \end{pmatrix}, \qquad \det G = 44593920.$$

Proposition 9.4 therefore gives the attained minimum

$$(0, -5)G^{-1}(0, -5)^T = \frac{25 \cdot 395136}{44593920} = \frac{35}{158}.$$

The single functional $L$ gives only the zero lower bound. The second reading detects inherited energy outside the later-family space; the joint matrix accounts for the overlap of the two readings. This test vector illustrates completeness of the invariant set at one finite scale.

## Glossary

The entries give the meaning of the notation, a small example or useful distinction, and a route back to its calculation. Terms such as measure, primitive and total variation have their usual mathematical meanings. Terms such as collision amplitude and parent shell name their particular roles in the energy account.

**Activation cutoff.** The boundary selecting which integer addresses are observed. With cutoff $S$, the native tail admits $n > S$, beginning at $S+1$. Moving the cutoff to $T > S$ removes the old readings on $S < n \le T$. It changes the observation even if the field is unchanged. Activation at $S+1$ admits the whole reading there, which need not be a clock reset (Section 10).

**Address.** A location where a field has a specified reading, such as a point of the periodic interval or an integer index. The observation determines how much mass that location receives. A grid address has mass $1/q$; an integer address in the native tail has mass $1/[n(n+1)]$. Under ordinary integration a single address has zero mass (Section 3).

**Adjoint receiver.** The detector moved back through the field map to source coordinates. If the field is $Ac$ and its detector is $d$, then $d^*Ac = (A^*d)^*c$. A weight matrix belongs inside this operation. The receiver's norm bounds all source directions; its pairing with the actual signed source is a further calculation (Section 24).

**Admissible periodic field.** A specified field that is piecewise $C^2$ on finitely many arcs of the periodic interval. On each closed piece, its value and first two derivatives extend continuously. The actual readings at boundaries are assigned separately. A field that vanishes except at one address is admissible when that address is included in the boundary set; admissibility does not require continuity across a boundary (Section 4).

**Alignment deficit.** The finite shortfall $\delta_{b,p}(m) = 2/p - \alpha_b(pm) = (p-2)/[p(pm-1)]$ for the specified aligned-tail count, with $p \ge 3$ dividing $b-1$ and $m$ supported by the primes of $b$. Its ratios approach ratios of supported integers under common scaling, with an explicit correction. The binary-field realization gives an energy difference for a particular observation; it is distinct from the carry field's sampling defect (Section 13).

**Atom.** In the observations used here, an address carrying positive mass by itself. If $\rho(\{b\}) = m > 0$, its contribution to energy is $m|f(b)|^2$. The symbol $\delta_b$ denotes unit mass concentrated at $b$. Thus the equally weighted grid is the measure $q^{-1}\sum_{b\in X_{q,\theta}} \delta_b$. Its atoms explain why changing one reading can change sampled energy while leaving the ordinary integral unchanged (Section 3).

**Base-supported integer.** A positive integer whose prime factors all divide a chosen base $b$. These integers form the nonnegative cone of its prime-exponent lattice. Bases thirty and sixty have the same support $\{2, 3, 5\}$. Support is a divisibility condition, not merely a bound on prime size (Section 13).

**Bernoulli kernels.** The fixed periodic functions $\psi(t) = \{t\} - 1/2$ away from integers, with $\psi(k) = 0$, and $\widetilde{B}_2(t) = \{t\}^2 - \{t\} + 1/6$. In boundary decomposition they weight value jumps, slope jumps and regular curvature. Their argument $qb - \theta$ records the location of a boundary within a sample cell. The first kernel has a jump and a specified midpoint reading; the second is continuous (Sections 4 and 5).

**Blind observation.** A specified detector record that vanishes on a nonzero source. The first event can vanish while a full return history recovers the source. Invertibility of the whole record and a uniform lower work bound are different requirements (Section 21).

**Boundary data.** The actual reading, the two one-sided traces and the derivative traces at a boundary. These are distinct data. For the opening sawtooth at zero, the left trace is 1, the right trace is $-1$ and the reading is 0. For energy calculations, first form $h = |f|^2$ and its boundary data. The remaining input to the decomposition is $h''_{\mathrm{reg}}$ between boundaries (Sections 4 and 5).

**Boundary localization.** Restricting the effect of a boundary change to the readings it crosses. In a finite register, changing $J$ columns costs the incidences in those columns, even when the rest of the register is large. This gives a budget before a smoothing width is chosen (Section 15).

**Carrier signal.** The sequence $u(n)$ modulated by an envelope in a moving-window field. The alternating carrier $(-1)^n$ has bounded partial sums, so its interaction can be controlled by variation of the weighted partner. This signal is distinct from the carrier modulus of an arithmetic kernel (Section 11).

**Carry count.** A sum of floor differences recording integer crossings at specified addresses. In the base-three table, the equal-digit positions $0, 4, 8$ give differences $0, 0, 1$ when $a = 1$, hence $C_3(1) = 1$. Subtracting $a/p$ centers the reading. The carry representation expresses the whole centered table as clock samples, making shared resets and inherited readings available to boundary decomposition (Section 25).

**Carry table.** A table of boundary crossings. A row fixes the multiplier $a$ and a column fixes an equal-digit position $n$. The entry is one when advancing $[na]_{p^2}$ by $a$ crosses the join of the $p^2$-address circle. Row sums give the carry counts; subtracting $a/p$ gives their centered readings (Section 1).

**Centering.** Subtracting a specified reference level from a reading. In the carry table the level is $a/p$, the proportional allowance for the selected positions. The centered reading records excess or shortfall. Its square contributes to the collision energy (Section 1).

**Chinese remainder theorem.** The rule combining congruences into one residue class. Conditions modulo $c_1, c_2$ are compatible when their residues agree modulo $(c_1, c_2)$; their common rows then form one class modulo $[c_1, c_2]$. A finite observation must still test whether that class meets its interval (Section 14).

**Clock.** A periodic fractional or sawtooth profile. The clock $\{u/j\}$ has period $j$, rises linearly between resets and reads zero at multiples of $j$. A centered clock $2c\psi(rx + \phi)$ has frequency $r$, phase $\phi$ and coefficient $c$; its jump is $-2c$. Its specified reset reading is the midpoint. The profile and normalization differ as well as the reset convention. Point-value replacement applies after fixing the surrounding profile (Sections 6 and 10).

**Collision.** Agreement between specified digits in a digit table. For fields, component contributions meet at a common observed address. For real readings $u, v$, the combined square contains $2uv$ in addition to their separate squares. At a shared reset, the combined jump and its interaction with the inherited field record that collision. The mixed term can be positive, negative or zero; the coincidence alone does not determine its sign (Sections 1, 3 and 6).

**Collision amplitude.** For centered clocks resetting at $b$, the combined coefficient $C_b = \sum_{j \in I_b} c_j$. Their total jump is $-2C_b$. Coefficients 1 and $-1$ cancel at a shared reset, whereas coefficients 1 and 1 reinforce. Taking the sum before its squared magnitude retains both possibilities. In a right-hand fractional-clock field, the analogous reset amplitude is the negative divisor sum of its source coefficients (Sections 6 and 10).

**Collision invariant.** The digit-agreement deviation $S_\ell(N) = C_N(b^\ell) - \lfloor (N-1)/b \rfloor$, which is determined by $N$ modulo $b^{\ell+1}$ when $N > b^{\ell+1}$ and $N$ is coprime to $b$. The agreement count can grow while this deviation stays fixed (Section 1).

**Collision matrix.** The matrix $H = G^{\mathrm{cont}} - G^{\mathrm{grid}}$ storing all pairwise mixed defects of a chosen component family. For amplitudes $z$, the total defect is $z^* H z$. The diagonal stores individual defects; off-diagonal entries retain interactions. Although each Gram matrix is positive semidefinite, their difference can have either sign. A change of coordinates reorganizes these interactions without supplying an estimate for them (Section 9). The continuous Gram's divisor factorization and the sampled Gram in the same coordinates are developed in Section 17.

**Collision register.** A finite table recording the components incident to each observed row, with their amplitudes and orientations. Its row is $\Delta(r) = \sum_\xi \sigma_\xi z_\xi I_\xi(r)$. Squaring this combined reading retains the interaction of every pair; the incidence kernel records their actual shared observations (Section 14).

**Collision table.** A record of specified digit comparisons. In the opening example, columns are remainders and rows are multipliers. An entry is one for matching first digits and zero otherwise. A row sum keeps the number of agreements while losing their positions (Section 1).

**Compatibility.** The existence of a common solution to arithmetic conditions before restricting the observation. The residues 4 modulo 7 and 1 modulo 8 are compatible, but their common class 25 modulo 56 misses $3 < r < 6$. Compatibility alone is not an observed collision (Section 14).

**Completed family.** In the divisor application, an old family whose source finishes before the taper begins. For odd $m$ with $8m \le S$, its field is the unchanged periodic profile $H_m$. It has zero period mean and an interval-sum bound $3m$. Screening uses this bound to control one interaction by $129m/S^2$; summing over many families must still account for their number and sizes (Section 27).

**Conserved mass.** The additivity of an observation under subdivision. If children partition a parent of mass $m$, their masses sum to $m$. Copying the same reading $c$ to those children preserves the energy $m|c|^2$ as well. Different child readings require their own weighted square. For the native observation, resolving the first two addresses gives masses $1/2, 1/6$ and remaining tail mass $1/3$; their sum is still one. Restricting to the tail selects a share of that mass (Sections 2.5 and 10.1).

**Constrained minimum.** The least energy obtainable by adding any field from a specified allowed space $V$ to an inherited field $f$. Invariant readings give lower bounds for this minimum. A complete collection detects every direction outside $V$ and determines the minimum through the joint matrix. A partial collection can miss energy, so a small lower bound need not mean that a small energy is attainable (Proposition 9.4). A general positive source Gram gives the measured least-work lift in Section 19.

**Cubic law.** The asymptotic $E_p = p^3 + O(p^2(\log p)^2)$ for the complete centered carry energy as the odd prime base grows. The secondary term identifies structure inside this error bound. The sampling defect compares the same finite energy with its continuous field (Section 1).

**Cyclic difference.** For forward shift $(Sx)_j = x_{j+1 \bmod L}$, the difference is $D = I - S$. It kills constant tables. The second difference $D^2$ has the same constant kernel on a cycle and can be inverted after its mean is supplied (Section 20).

**Defect.** The signed difference between the energies of one field under two observations. The periodic convention is $D_{q,\theta}[f] = \int_0^1 |f|^2\,dx - Q_{q,\theta}(|f|^2)$. A positive value means the integral is larger; a negative value means the samples carry more energy. The square is taken before averaging: $Q(|f|^2)$ generally differs from $|Qf|^2$. A defect is quadratic in the field and can be zero (Sections 3 and 4).

**Dilation.** Composition with an integer multiplication of the periodic argument, as in $f(x) \mapsto f(rx)$. Dilation preserves the continuous integral of a periodic profile but can change which addresses a finite grid visits. On a six-point grid, doubling the argument visits $0, 1/3, 2/3$ twice each. The greatest common divisor of the frequency and grid size records that repetition (Proposition 9.2).

**Discrete curvature.** The second difference $D^2x$ of a cyclic reading table. It records changes of the first differences and loses the constant component. It is distinct from the ordinary second derivative between continuous boundaries (Section 20).

**Divisor coordinates.** Coordinates formed by summing source coefficients over divisibility relations. For harmonic frequencies, reset masses are $m_d = \sum_{d|h} c_h$, whereas primitive work coordinates are $q_d = \sum_{d|h} c_h/h$. The weighting and direction of the divisor sum are part of the definition (Section 17).

**Dual field.** The source $u = G^{-1}r$ representing the covector $r^*$ in a positive work form $G$. Its pairing $u^*Gc = r^*c$ reproduces the detector. Its work $r^*G^{-1}r$ gives the detector's squared norm in that measured geometry (Section 24).

**Energy.** Squared magnitude integrated against the observation, $E_\rho[f] = \int |f|^2 \, d\rho$. It is nonnegative for a positive observation. With masses $1/2, 1/2$, readings $1, 3$ have energy 5. A sum of fields also contains mixed energy, so energies cannot ordinarily be added component by component. The energy is a mean square only when the observation has total mass one (Section 3).

**Field.** A function supplying a specified reading at each address. The field includes its assigned boundary values, even where changing a value would leave an ordinary integral unchanged. Separating the field from the observation makes two operations distinguishable: changing a reading and changing the mass assigned to it. A moving arithmetic family can change both at the same step (Section 3).

**Fixed carrier.** An arithmetic parameter retained while the observation cutoff changes. For $R_q(a,b)$ the carrier modulus is $q$; $\mathcal{A}_q(M)$ changes the square cutoff $M$ without changing $q$. A bound at a smaller prime generally changes the carrier as well, so it cannot automatically be used for a shorter prefix of the old kernel (Section 16).

**Fourier mode.** A periodic oscillation $e^{2\pi ikx}$, with integer frequency $k$. Averaging it on the unshifted $q$-point grid gives zero unless $q$ divides $k$, in which case the average is one. The boundary proof uses this selection rule after expressing the field's jumps in its Fourier coefficients. An offset changes the phase of a surviving mode, producing the shifted Bernoulli kernels (Theorem 5.1).

**Full record.** The complete source or reading table before identifying directions unseen by the current observation. Refinement may make an old null direction visible. The old quotient suffices for a refined detector precisely when its nullspace remains unseen after transport (Section 22).

**Grid offset.** The parameter $\theta$ in the sample addresses $(a+\theta)/q$, for $0 \le a < q$. It is measured in cells: $\theta = 1/2$ shifts the grid by $1/(2q)$, not by half the whole interval. A boundary $b$ is sampled precisely when $qb - \theta$ is an integer. Changing the offset can therefore remove a point contribution even when the continuous energy is unchanged (Section 4).

**History energy.** Work accumulated or averaged over a specified complete sequence of returns. With $R^L = I$, the averaged history Gram is $L^{-1}\sum_{j=0}^{L-1}(R^j)^* H R^j$. It is return-invariant even if one event's observation $H$ is not (Proposition 21.1).

**Inherited interaction.** The cross term $2\operatorname{Re} K_\rho(f,g)$ between an old field $f$ and an addition $g$. It is measured under the observation used for the new combined field. At one real reading it is $2fg$: an addition of $-1$ to an inherited reading 1 gives cross term $-2$, which combines with the new square 1 to remove the old energy (Proposition 3.1).

**Inserted seam.** A new cut at which refinement and one step of motion can give different readings. In the two-to-four subdivision, the connection sends $(a,b)$ to $(a-b,0,b-a,0)$. Its transported copies can overlap and interact in a longer history (Section 22).

**Intersection kernel.** The weighted observation shared by two components: $K_v(\xi,\eta)=\sum_r v(r)I_\xi(r)I_\eta(r)$. It includes the common residue, the actual interval and its row weights. These entries form a Gram matrix of incidence vectors (Section 14).

**Invariant reading.** A linear combination $L$ of readings that vanishes on every allowed addition $h\in V$. Consequently $L(f+h)=L(f)$, and this retained value forces an energy lower bound. For example, a second difference vanishes on every affine addition. Several such readings may share addresses; their separate lower bounds cannot generally be added without accounting for the overlap (Theorem 9.3).

**Inverse cost.** The norm by which a reconstruction map can amplify an input error. The inverse cyclic second difference has norm $1/[4\sin^2(\pi/L)]$, of order $L^2$. Squaring that norm for an energy allowance costs order $L^4$ (Proposition 20.1).

**Joint matrix.** The matrix $G=AW^{-1}A^*$ for independent invariant readings arranged as rows of $A$, with positive diagonal observation weights $W$. Its off-diagonal entries measure overlap between readings in the reciprocal weights. For retained values $b=Af$, the lower bound is $b^*G^{-1}b$. It is the constrained minimum when $\ker A=V$ (Proposition 9.4). The overlapping-second-difference problem (Problem 19) shows why separate bounds can double-count energy.

**Jordan weight.** The positive arithmetic weight $J_2(n)=n^2\prod_{\ell\mid n}(1-\ell^{-2})$, with the product over primes. It counts primitive pairs modulo $n$ and gives the continuous work $J_2(n)/3$ of $g_n$ (Proposition 17.2).

**Jump.** Right trace minus left trace, $[h]_b=h(b+)-h(b-)$. A fractional clock resetting from one to zero has jump $-1$. Its actual value at the boundary remains a separate choice: replacing that reading by $1/2$ changes no trace and hence no jump. For energy decomposition, the required jumps are those of $h=|f|^2$ and its first derivative (Section 4).

**Least-work lift.** The source of smallest work with a prescribed linear record. For positive $G$ and independent rows $U$, it is $Ty$, where

$$T = G^{-1}U^{*}(UG^{-1}U^{*})^{-1}.$$

The minimum is unique on the positive source space or observable quotient (Proposition 19.2).

**Literal refinement.** A specified copying of the full source to finer addresses. The example $R(a, b) = (a, a, b, b)$ subdivides each old cell, while the cycle cover $C(a, b) = (a, b, a, b)$ repeats the complete cycle. Both preserve normalized work but have different one-step motion laws (Section 22).

**Local energy.** The energy on a specified subinterval or subset of addresses. For an affine field on $(0, h)$ it is $h(a + bh/2)^2 + b^2h^3/12$, recording mean height and slope separately. A shrinking interval changes the cost of reading a derivative (Section 24).

**Localization.** Expressing a complete energy change as a finite calculation plus a proved allowance for the omitted tails. The old tail energy supplies the lower allowance and the new tail energy the upper one. A finite calculation may have the wrong sign while the resulting interval remains valid. Localization becomes effective when field envelopes and remaining observation mass make that interval sufficiently short (Theorem 10.4).

**Log-phase difference.** A translation difference $\Delta_a f(z) = f(z) - f(z + a)$ in logarithmic scale. With $a = \log q$, it compares physical scales $s$ and $qs$. Products of such differences produce complete signed packets; they are distinct from the logarithmic prefix transform (Section 23).

**Logarithmic prefix transform.** The operation

$$\mathcal{K}(N) = -\sum_{d \le N} \mu(d) \log d \, \mathcal{A}(\lfloor N/d \rfloor)$$

on square prefixes of a fixed kernel. For a dilation-invariant kernel it equals $\sum_{m \le N} P(m)\Psi(N/m)$, combining primitive shells with the prime-power count (Theorem 16.2). Translation differences in logarithmic phase are a separate operation, introduced in Section 23.

**Mean record.** The constant component retained alongside differences to reconstruct a table. One mean is needed for each disjoint cycle. It is part of the source data and can contribute both its own energy and mixed work under a nonuniform detector (Section 20).

**Measured projection.** A projection orthogonal in the specified positive work form. For the least-work lift it is $\Pi = TU$ and satisfies $\Pi^{*}G = G\Pi$. Ordinary coordinate axes need not display its orthogonality (Section 19).

**Midpoint reading.** The arithmetic mean $(f(b-) + f(b+))/2$ of a field's two traces. The average of their squared magnitudes exceeds the squared midpoint by $|[f]_b|^2/4$. This produces a nonnegative point contribution at sampled boundaries. An actual right-hand reading requires a further replacement term, which includes interaction with the midpoint. Its correction need not be positive (Corollary 5.2 and Section 10).

**Mixed energy.** The form $K_\rho(f, g) = \int f\overline{g}\, d\rho$. Twice its real part is the cross term in $E_\rho[f + g]$. It is linear in the first field and conjugate-linear in the second, and interchanging the fields conjugates the result. These are sesquilinearity and Hermitian symmetry. Subtracting the mixed energies under two observations gives the mixed defect, which need not be positive (Sections 3 and 9).

**Möbius coefficient.** The coefficient $\mu(r)$ is zero when a prime square divides $r$. Otherwise $\mu(r) = (-1)^k$, where $k$ counts the distinct prime factors, with $\mu(1) = 1$. Thus $\mu(3) = -1$, $\mu(15) = 1$ and $\mu(9) = 0$. These signed coefficients join the divisor families. Their divisor identity permits inversion between family data and reset data; their signs alone do not bound the resulting energy (Sections 16 and 27).

**Modular inverse.** An integer $\bar{a}$ satisfying $a\bar{a} \equiv 1 \pmod{k}$; it exists precisely when $\gcd(a, k) = 1$. For example, 25 is its own inverse modulo four, while 50 has no inverse modulo four. The half-cell calculation therefore separates odd and even moduli before making inverse substitutions. A bar in that calculation refers to the modulus of its summand (Section 26).

**Native weights.** The prescribed masses $w_n = 1/[n(n+1)] = 1/n - 1/(n+1)$. They are the reciprocal-cell integrals $\int_n^{n+1} u^{-2}\,du$, so the tail beyond $S$ has total mass $1/(S+1)$. Each later address receives less mass, but every address in the infinite tail remains present. Summation by parts must retain the variation of these masses as well as that of the field (Section 10).

**Observable quotient.** The source space with directions unseen by a specified full observation identified with zero. Its work becomes positive definite. A separate detector descends to this quotient only if it also vanishes on those directions (Section 19).

**Observation.** A finite positive measure specifying which addresses are read and with what mass. It covers ordinary integration, a finite weighted table and a summable infinite tail. The word measure records the assignment of mass, not an additional field. For an observation of total mass $M$, the constant field one has energy $M$. Changing the observation can change energy without changing a single reading (Section 3).

**Operator norm.** The largest ratio $\|Mv\|/\|v\|$ over nonzero input vectors, using the specified vector norm. It bounds every input direction. A prescribed quadratic reading $v^T M v$ may be small even when the operator norm is large. The arithmetic interior construction makes that distinction explicit (Proposition 18.1).

**Packet fringe.** Members left at a cutoff where a complete scale packet has been clipped. Their readings must be restored or bounded separately. A missing member can destroy the cancellation of a constant observation (Section 23).

**Packet kernel.** The compact profile produced by differences of a step, or its use in a convolution with a source measure. Two differences give a kernel of zero mass and generally nonzero first moment. Its observation is controlled by a first derivative (Section 23).

**Parent shell.** The addresses $S < n \le T$ removed when a native cutoff moves from $S$ to $T$. Their entire old energy enters the change with a minus sign. If the old field is a sum of families, the shell term contains their combined square and every cross term. Replacing it by a sum of family energies would alter the account (Proposition 10.3).

**Point-value replacement.** Changing assigned readings while retaining the surrounding profile. At an atom of mass $m$, replacement of $f(b)$ by $g(b)$ changes sampled energy by $m(|g(b)|^2 - |f(b)|^2)$. It changes an integral-minus-sample defect by the opposite amount when the integral is unchanged. At an unobserved address the correction is zero. This rule restores actual readings after a convenient midpoint calculation (Proposition 4.1).

**Polarization.** Recovering interaction from the energy of a sum. Here the basic expansion is $D[f+g] = D[f] + D[g] + 2\,\mathrm{Re}\,K(f,g)$, where $K$ is the mixed defect. Real fields $f$ and $-f$ illustrate its role: their sum has zero defect even when each separate defect is nonzero. The collision matrix organizes the same expansion for many components (Section 9).

**Polynomial projection.** A component selector expressed as a polynomial in a return operator. Finite Fourier sums separate the eigenspaces of a finite-order return. Their measured orthogonality uses an invariant history form (Section 21).

**Prime-exponent lattice.** Integer coordinates recording the powers of a fixed set of primes in a positive rational number. Multiplication adds coordinates. For support $\{2, 3, 5\}$, the octave, fifth and major third have coordinates $(1, 0, 0)$, $(-1, 1, 0)$ and $(-2, 0, 1)$. The alignment deficit supplies a digit-count coordinate on the supported positive integers (Section 13).

**Prime packet.** A complete family indexed by divisors of a squarefree product, with signed coefficients supplied by the chosen difference. For primes two and three, the four scale readings have coefficients $1, -1, -1, 1$ at $s, 2s, 3s, 6s$. Its cutoff members and observation weights remain part of an application (Section 23).

**Prime-counting function.** Here the weighted prime-power count $\Psi(x) = \sum_{\ell^j \le x} \log \ell$, where $\ell$ runs over primes and $j \ge 1$. For example $\Psi(4) = 2\log 2 + \log 3$. The logarithmic transform gives a finite identity involving this quantity; estimating its error requires further control of the signed recovery (Section 16).

**Primitive.** A cumulative sum, playing the role of an antiderivative for a sequence. Anchored at $a$, it is $T_a(n) = \sum_{k=a}^{n} u(k)$ with $T_a(a-1) = 0$. For $u = (1, -1, 1, -1)$ its readings are $1, 0, 1, 0$. A small primitive records cancellation in $u$. Summation by parts pairs it with differences of the weighted partner, retaining endpoints (Section 8).

**Primitive field.** The midpoint-clock combination $g_n = \sum_{h|n} h\mu(n/h) f_h$. These fields are orthogonal for continuous work with weights $J_2(n)/3$. They need not be orthogonal, or independent, after a finite sampling observation (Section 17).

**Primitive shell.** The signed kernel sum over coprime pairs whose maximum coordinate is $m$. Each such pair has $\lfloor M/m \rfloor$ common multiples in a square cutoff $M$. Primitive means coprime coordinates; it does not mean a prime shell index, and differs from a cumulative primitive (Proposition 16.1).

**Quadrature.** Estimating an integral by a weighted sum of sampled readings. The periodic grid rule $Q_{q,\theta}$ is one such operation. Its discrepancy is linear in the sampled function; applying that discrepancy to $|f|^2$ gives the energy defect. A quadrature rate for a fixed function can depend on its jumps and curvature, which must be counted again when the field grows (Sections 5 and 12).

**Quantitative recovery.** Reconstructing a quantity while controlling the size of the error or bound carried through the reconstruction. A triangular identity can be exact even when an absolute-value induction enlarges its allowance. The prime-shell example has coefficient $\sqrt{5}/2 > 1$ at square-root scale (Section 18).

**Receiver norm.** The size of a transported detector in a specified norm. In positive work $G$, the squared dual norm is $r^* G^{-1} r$. In a signed-square expansion the Euclidean squared norm can supply the diagonal, but the mixed covariance still remains (Section 24).

**Reciprocal neutrality.** The condition $\sum_j q(j)/j = 0$ for a finite fractional-clock source. Each clock has slope $1/j$ between resets, so this signed sum is the slope of their combined field. Neutrality makes that field constant between integer addresses, leaving all changes at resets. For $p(u) = \{u\} - 2\{u/2\}$, the source sum is $1 - 2/2 = 0$ (Section 10).

**Reciprocal phase transfer.** The identity $e(A\bar{r}/c) = e(-A\bar{c}/r)e(A/(rc))$ for coprime $r, c > 1$ and integer $A$, with $e(x) = \exp(2\pi i x)$. It changes the modulus of an inverse while retaining a reciprocal correction. That correction remains part of any subsequent phase estimate (Proposition 15.3).

**Refinement.** A specified change of resolution or scale. Refining a grid changes the sampling observation. Increasing the number of clocks changes the field as well; moving its activation cutoff changes the observed domain. Each change has its own term in the energy identity. Statements about refinement therefore specify which objects vary and which stay fixed (Sections 9 and 10). For unity, subdivision resolves the same total mass, while admitting new clocks enlarges the space available to represent the fixed target (Section 2.5). Transport through a source map also requires the complete record when an old null direction can become visible (Section 22).

**Refinement connection.** The map $E = TR - RS$ measuring the difference between advancing after refinement and refining after advance. Its exact history is the sum of its transported copies in (65). The identity retains accumulation and interactions; it does not assert their decay (Section 22).

**Regular curvature.** The ordinary second derivative $h''_{\mathrm{reg}}$ on the open pieces between boundaries. Value jumps and slope jumps are recorded separately in the boundary formula. For a piecewise linear field, the field itself has zero second derivative on each piece, but its squared magnitude generally has nonzero second derivative. Energy decomposition requires the latter, because its input is $h = |f|^2$ (Section 5). Cyclic second differences act on a finite table and require a separate mean record (Section 20).

**Reset.** An address where a fractional clock returns to the start of its cycle. The actual fractional-part reading is zero, its right-hand value. Clocks of lengths dividing $n$ reset together at integer address $n$. Their signed coefficients are combined before forming the energy change. A reset of some clocks does not remove the inherited reading of all the others (Sections 6 and 10).

**Reset mass.** For midpoint harmonic clocks, $m_k(c) = \sum_{k|h} c_h$, the coefficient sum resetting at any reduced boundary of denominator $k$. Its signed field jump is $-2m_k$. These masses determine the whole finite source by inversion (Section 17).

**Reset selector.** The map $P_\sigma = Z^{-1}\operatorname{diag}(\sigma)Z$ that masks reset data and reconstructs the selected source. It is a projection for a zero-one mask but generally not an orthogonal one. Deleting source entries is a different operation (Proposition 17.1).

**Residual.** The terms remaining after a specified contribution has been separated from an identity. For a periodic defect, $R = D - P$ remains after the sampled point contribution $P$. For native work, the remainder after short-cycle screening includes the parent shell and unscreened tail interactions. A residual keeps its sign. Naming it does not imply that it is small or tends to zero (Sections 9 and 28). In approximation of unity, the residual is the field $r_N = \mathbf{1} - p_N$; its complete squared error is $E_0[r_N]$. The field and its scalar error have different roles (Section 10.1).

**Return operator.** A specified advance of the source or its full record. Following a detector through its complete finite orbit defines a history observation. A single-event observation need not be an isometry for that advance (Section 21).

**Screening.** Control of an interaction by cancellation within an old completed cycle. Its field has zero mean, so a constant partner cancels over a full period. For a varying weighted partner, summation by parts uses the old field's bounded primitive and the partner's variation. The native estimate $129m/S^2$ controls each interaction; the sum also depends on how many cycle lengths $m$ are included (Theorem 27.2).

**Signed covariance.** The deterministic mixed term $2\sum_{j<k} s_j s_k r_j r_k$ in the square of a signed reading. The diagonal alone cannot determine it. No independence of the signs is included in the term's definition (Section 24).

**Signed work.** The change in observed energy under a specified modification of field and observation. For an addition $g$ under a fixed observation it is $2\operatorname{Re} K_\rho(f, g) + E_\rho[g]$. A moving cutoff also contributes the removed parent energy. Keeping these terms joined permits negative interaction to offset the new square. Bounding their absolute values separately can lose that cancellation (Proposition 10.3).

**Smoothing width.** The number of boundary columns over which a sharp cutoff is replaced by a smooth transition. Wider transitions change more readings and have smaller derivative norms. Balancing costs $J$ and $p/\sqrt{J}$ selects $J \asymp p^{2/3}$ under the stated smooth-row hypothesis (Proposition 15.2).

**Source.** A coefficient table for a clock field. Each entry $q(j)$ gives the amplitude of the clock $\{u/j\}$, whose period is $j$. This $q(j)$ is distinct from the earlier scalar grid size $q$. Its reciprocal sum controls the slope, and its divisor sum $c(n) = \sum_{j|n} q(j)$ controls the simultaneous reset. Sources therefore convert coefficient data into local field changes (Section 10). For unit-circle harmonic clocks the coefficients are $c_h$, and reset masses instead sum over multiples of the denominator, $m_k = \sum_{k|h} c_h$ (Section 17).

**Tail.** All readings beyond a chosen finite endpoint. For native weights, the tail beyond $N$ has mass $1/(N+1)$. If $|p(n)| \le A$ there, its energy is at most $A^2/(N+1)$. When $A$ depends on an outside scale, the endpoint must grow sufficiently fast to control that dependence. A finite head together with this allowance encloses the full observation (Theorem 10.4).

**Taper.** A factor that reduces a source profile before its terminal cutoff. The divisor application uses $\tau_S$, equal to one through $S$, decreasing linearly to zero at $2S$, and zero beyond. Taking the difference of a tapered profile also differentiates the taper discretely. Its resulting coefficients, including the terminal one, must remain in the source (Section 27).

**Total variation.** The supremum, over ordered partitions, of the sum of absolute successive changes in a function. For a monotone function it is the absolute endpoint difference: $x^{-2}$ has variation $3/4$ on $[1, 2]$. In weighted primitive estimates, variation measures how strongly the partner can disturb the carrier's cancellation. Both changing weights and changing field values contribute (Section 11).

**Trace.** A one-sided limiting reading, written $f(b-)$ or $f(b+)$. At the periodic origin, the left trace comes from addresses near one and the right trace from addresses near zero. The actual value $f(b)$ may equal either trace, their midpoint or another assigned value. Traces describe the surrounding profile; the observation determines whether the assigned point reading also contributes (Section 4).

**Transported density.** The factor carried by a continuous measure under a coordinate change. For the increasing, piecewise affine map $v = \Theta(u)$ used here, it is $(\Theta^{-1})'(v)$. Moving a boundary while fixing sample points can still change the continuous reading if this factor is omitted (Section 19).

**Uniform estimate.** A bound with constants independent of the varying field or scale over the stated family. The assertion $|D_q[f]| \le C_f/q$ for each fixed $f$ does not by itself control $D_q[f_q]$: the constants $C_{f_q}$ may grow with $q$. A uniform proof bounds the relevant amplitudes, jumps, primitive sizes and variation across the whole family before taking the limit (Section 12).

**Unity.** The constant field $\mathbf{1}(x) = 1$, used as a fixed target through changing resolutions and approximation spaces. Under an observation of mass one its energy is one. Under the native tail observation it has energy $E_S[\mathbf{1}] = 1/(S+1)$, the mass actually retained. An approximation $p_N$ leaves residual $\mathbf{1} - p_N$. For example, twice the period-two fractional clock equals unity at odd addresses and leaves the even addresses unresolved. Their complete error mass is $1 - \log 2$. Unity, total mass and the energy of a general field are distinguished in Sections 2.5 and 10.1.

**Vanishing moment.** An identity $\int z^k K(z)\, dz = 0$ for a specified kernel. It cancels a polynomial observation of degree $k$. The number of such identities must be established for the actual kernel; it differs from the degree of input profiles annihilated by a translation difference (Section 23).

**Visibility bound.** A quantitative lower comparison between observed work and a prescribed source work. Positive visibility at each scale need not have a uniform positive lower bound. The example with detector gain $\delta$ has lower constant $\min(1, \delta^2)$ (Section 21).

**Work form.** The positive quadratic form measuring a source under an observation. For component map $V$ and address weights $W$, its matrix is $G = V^* W V$. The off-diagonal entries retain component interactions (Section 19).

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# The arithmetic of what remains.

We measure by making distinctions. A chosen unit sets the spacing of a grid. Readings give us a finite account of something we describe as continuous.

A limit can reveal a law while leaving its finite correction unresolved. A table can be exact while leaving the behavior beyond its scale unknown. Understanding their relation means following what each description keeps.

*Collision Calculus* studies that relation through energy. Boundaries, interactions and cancellation give the difference between a finite reading and a continuous whole a structure of its own. Worked examples, figures and proofs follow it from simple digit agreements into growing arithmetic fields.

A fuller understanding asks us to follow the relation in both directions.

## How does a continuous whole find expression in the finite?



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