MAGNITUDE DECORRELATION IN THE COLLISION SPECTRUM

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ABSTRACT. For primitive odd characters modulo the square of a growing odd prime base b, we prove that the Pearson correlation of the short diagonal-sum magnitude and the generalized Bernoulli magnitude is $O((\log\log b)^{-1/4})$, and hence tends to zero. The proof combines a prime-square adaptation of Harper's short-sum argument with direct L-value moment and variance bounds. The squared magnitudes retain a different relation. Their mixed moment is normalized collision energy, whose ratio to the product of the marginal second moments tends to three by the cubic law. The secondary term supplies the logarithmic correction. Thus magnitude decorrelation coexists with a persistent excess in the squared weights.

1. Two Magnitudes

The revised *Collision Spectrum* [5] ends with a question. Beyond base five, its exhaustive finite ledger records a positive relation between the short diagonal sum and the Bernoulli magnitude, but proves no limiting law as the prime base grows. We resolve this magnitude question for Pearson correlation. Its limit is zero.

At base five, the two magnitudes are proportional in every primitive odd character. At base seven, they are not. The correlation remains positive at each tested base, falling from 1 at base five to approximately 0.6753 at base 71. Proving that it eventually tends to zero requires control of the first magnitudes, which the collision-energy identity alone does not supply.

The squared weights retain a limiting ratio of three, as the collision-energy theorems make precise. The magnitude correlation vanishes while this excess persists.

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The question begins with two finite sums. Let $b \geq 5$ be an odd prime, let $m = b^2$, and write $\mathcal{P}_b$ for the primitive odd characters modulo m. Each character is extended by zero outside the units. Put

$$T_b(\chi) = \sum_{k=1}^{b-1} \overline{\chi}(k), \qquad B_b(\chi) = \frac{1}{b^2} \sum_{a=1}^{b^2-1} a\overline{\chi}(a).$$

The first sum reads one base-length interval. The second reads the complete residue system with a linear weight. The classical identity relating Bernoulli numbers to L-values gives, for primitive odd characters,

$$|B_b(\chi)| = \frac{b}{\pi}|L(1,\chi)|. \tag{1}$$

We use the functional-equation convention in [4].

For $n \in \{0, \ldots, b^2 - 1\}$, the equality of its two base-b digits selects the diagonal

$$G_b = \{r(b+1) \mid 0 \leq r < b\}.$$

The corresponding boundary sum is

$$D_b(\chi) = \sum_{n \in G_b} \big(\overline{\chi}(n+1) - \overline{\chi}(n)\big).$$

The diagonal reduction in [5, Lemma 3] gives

$$D_b(\chi) = -2\overline{\chi}(b+1)T_b(\chi), \qquad |D_b(\chi)| = 2|T_b(\chi)|. \tag{2}$$

Thus the short sum retains the full diagonal magnitude.

Normalize the two magnitudes by

$$X_b(\chi) = \frac{|T_b(\chi)|}{\sqrt{b}}, \qquad Y_b(\chi) = \frac{\pi|B_b(\chi)|}{b} = |L(1,\chi)|. \tag{3}$$

For a function f on $\mathcal{P}_b$, write

$$\mathbf{E}_b f = \frac{1}{N_b} \sum_{\chi \in \mathcal{P}_b} f(\chi), \qquad N_b = \frac{(b-1)^2}{2}.$$

Both members of every conjugate pair are counted. Positive rescaling does not change Pearson correlation, so the correlation of X_b, Y_b is the one recorded for $|T_b|, |B_b|$.

The magnitude correlation is

$$r_b = \frac{\mathbf{E}_b(X_b Y_b) - \mathbf{E}_b X_b \, \mathbf{E}_b Y_b}{\sqrt{\mathrm{Var}_b(X_b)\,\mathrm{Var}_b(Y_b)}}$$

where the two variances are positive.

Theorem 1 (Magnitude correlation)**.** *For all sufficiently large odd prime bases, r_b is defined and*

$$|r_b| \ll (\log\log b)^{-1/4}.$$

In particular, $r_b \to 0$ along the odd primes.

The proof is completed in Section 8. It uses a short-sum estimate obtained from Harper's argument [1], together with two direct bounds on the L-value magnitudes. No hypothesis on zeros of L-functions is used.

Exact quadratic moments over odd characters, including twists by a character value, belong to the established $L(1,\chi)$ moment literature. Louboutin gives such formulas [2]; see also the formulation and extension in Lee and Lee [3]. The calculations below give direct primitive-odd prime-square specializations and use the twist at two to bound the variance of $|L(1,\chi)|$. The fractional-moment mechanism is Harper's. We verify its comparison conditions for the interval $1 \le n < b$ at modulus b^2 and convert the resulting estimate into the Pearson bound. The mixed-square asymptotics use the separate collision-energy theorems [6, 7].

2. The Marginal Second Moments

The second moments have exact values before any asymptotic estimate is used. Character orthogonality supplies both of them.

Lemma 2 (Primitive odd kernel)**.** *For units u, v modulo b^2,*

$$\sum_{\chi\in\mathcal{P}_b} \chi(u)\overline{\chi}(v) = \frac{b(b-1)}{2}\big(\mathbf{1}_{u\equiv v\ (b^2)} - \mathbf{1}_{u\equiv -v\ (b^2)}\big) - \frac{b-1}{2}\big(\mathbf{1}_{u\equiv v\ (b)} - \mathbf{1}_{u\equiv -v\ (b)}\big).$$

Proof. The odd-character projector at either modulus $q = b, b^2$ gives

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(u)\overline{\chi}(v) = \frac{\varphi(q)}{2}\big(\mathbf{1}_{u\equiv v\ (q)} - \mathbf{1}_{u\equiv -v\ (q)}\big).$$

The imprimitive odd characters modulo b^2 are precisely the lifts of the odd characters modulo b. Subtract their projector. Setting $u = v = 1$ also gives $N_b = (b-1)^2/2$. $\square$

Proposition 3 (Exact marginal moments)**.** *For every odd prime $b \ge 5$,*

$$\mathbf{E}_b X_b^2 = 1, \qquad \mathbf{E}_b Y_b^2 = \nu_b := \frac{\pi^2}{6}\left(1 - \frac{1}{b^2}\right). \tag{4}$$

Equivalently,

$$\mathbf{E}_b|T_b|^2 = b, \qquad \mathbf{E}_b|B_b|^2 = \frac{b^2-1}{6}.$$

Proof. Expand $\sum_{\chi\in\mathcal{P}_b} |T_b(\chi)|^2$ and apply Lemma 2. Among $1 \le u, v < b$, the positive congruence modulo b^2 holds exactly when $u = v$. The negative congruence never holds, since $0 < u + v < b^2$. Modulo b, the $b-1$ positive pairs $u = v$ and the $b-1$ negative pairs $u + v = b$ cancel. Hence

$$\sum_{\chi\in\mathcal{P}_b} |T_b(\chi)|^2 = \frac{b(b-1)^2}{2} = bN_b.$$

For an odd modulus q, the odd projector gives the complete Bernoulli square sum

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left| \frac{1}{q} \sum_{a\in\mathcal{U}_q} a\overline{\chi}(a) \right|^2 = \frac{\varphi(q)}{2q^2} \sum_{a\in\mathcal{U}_q} (2a^2 - qa), \tag{5}$$

where $\mathcal{U}_q$ uses the representatives $1 \le a < q$ coprime to q. At $q = b^2$, subtracting the multiples of b from the full power sums yields

$$\sum_{a\in\mathcal{U}_{b^2}} (2a^2 - b^2 a) = \frac{b^2(b-1)(b^3-2)}{6}.$$

The full odd-character Bernoulli square sum is therefore

$$\frac{(b-1)^2(b^3-2)}{12b}.$$

If χ is lifted from an odd character ψ modulo b, then

$$\frac{1}{b^2}\sum_{r=1}^{b-1}\sum_{t=0}^{b-1} (r+bt)\overline{\psi}(r) = \frac{1}{b}\sum_{r=1}^{b-1} r\overline{\psi}(r).$$

Here the constant term vanishes because $\sum_r \psi(r) = 0$. The imprimitive contribution is consequently the odd-character square sum at modulus b, namely

$$\frac{(b-1)^2(b-2)}{12b}.$$

Subtracting gives

$$\sum_{\chi\in\mathcal{P}_b} |B_b(\chi)|^2 = \frac{(b-1)^3(b+1)}{12}.$$

Divide by N_b and apply (1). $\square$

Proposition 4 (The base-seven obstruction)**.** *There is no constant c such that $|D_7(\chi)| = c|B_7(\chi)|$ for every $\chi \in \mathcal{P}_7$.*

Proof. If such a constant existed, Proposition 3 and (2) would give

$$c^2 = \frac{\mathbf{E}_b|D_7|^2}{\mathbf{E}_b|B_7|^2} = \frac{28}{8} = \frac{7}{2}.$$

The residue 3 generates the units modulo 49. Choose the character with $\overline{\chi}(3) = \zeta$, where ζ is a primitive fourteenth root of unity. It is odd, and it is primitive modulo 49. Grouping the powers of 3 by their exponents modulo 14 gives

$$T_7(\chi) = Q(\zeta), \qquad B_7(\chi) = H(\zeta),$$

where

$$Q(z) = 1 + 2z + z^{10} + z^{12} + z^{13}, \qquad H(z) = \sum_{j=3}^{9} z^j.$$

The required proportionality would imply $8|Q(\zeta)|^2 - 7|H(\zeta)|^2 = 0$. Use $\overline{\zeta} = \zeta^{-1}$ and reduce modulo $\Phi_{14}(z) = z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$. Direct multiplication gives

$$8|Q(\zeta)|^2 - 7|H(\zeta)|^2 = 4\big(3 - 2\zeta^2 + 3\zeta^3 - 3\zeta^4 + 2\zeta^5\big).$$

The polynomial in parentheses is nonzero and has degree five. It cannot vanish at a primitive fourteenth root, whose minimal polynomial has degree six. This contradicts the proposed constant. ☐

3. The Joint Square Mass

The product of the squared magnitudes is also controlled, through the collision energy. We give the finite normalization explicitly.

For a unit representative $1 \le a < b^2$, set

$$S_b(a) = -1 - \left\lfloor \frac{a}{b} \right\rfloor + \sum_{n \in G_b} \left(\left\lfloor \frac{(n+1)a}{b^2} \right\rfloor - \left\lfloor \frac{na}{b^2} \right\rfloor \right).$$

For $1 \le s < b$, subtract the mean within its reduction fiber,

$$\mu_b(s) = \frac{1}{b} \sum_{\substack{a \in \mathcal{U}_{b^2} \\ a \equiv s\ (b)}} S_b(a), \qquad S_b^\circ(a) = S_b(a) - \mu_b(a \bmod b).$$

The centered collision energy is

$$E_b = \sum_{a \in \mathcal{U}_{b^2}} |S_b^\circ(a)|^2.$$

Proposition 5 (Exact mixed square moment)**.** *For every odd prime* $b \geq 5$*,*

$$\mathbf{E}_b(X_b^2 Y_b^2) = \frac{\pi^2 E_b}{2b^2(b-1)}. \tag{6}$$

Consequently the ratio

$$R_b := \frac{\mathbf{E}_b(X_b^2 Y_b^2)}{\mathbf{E}_b X_b^2 \, \mathbf{E}_b Y_b^2} = \frac{3E_b}{(b-1)(b^2-1)} \tag{7}$$

is determined by the finite collision table.

Proof. With normalized Fourier coefficients, the spectrum factorization and Parseval identity of [5, Theorems 5 and 9] give

$$\widehat{S}_b^\circ(\chi) = -\frac{B_b(\chi)\overline{D_b(\chi)}}{b(b-1)}, \qquad \sum_{\chi \in \mathcal{P}_b} |\widehat{S}_b^\circ(\chi)|^2 = \frac{E_b}{b(b-1)}.$$

All omitted coefficients vanish by reflection or fiber centering. Using (1) and (2), we obtain

$$\sum_{\chi \in \mathcal{P}_b} |T_b(\chi)|^2 |L(1,\chi)|^2 = \frac{\pi^2(b-1)}{4b} E_b.$$

Division by bN_b proves (6). Proposition 3 then gives (7). □

Corollary 6 (The growing-base square ratio)**.** *Along odd prime bases,*

$$\mathbf{E}_b(X_b^2 Y_b^2) = \frac{\pi^2}{2} - \frac{(\log b)^2}{2b} + O\left(\frac{(\log(2b))^2}{b \log\log(3b)}\right), \tag{8}$$

$$R_b = 3 - \frac{3}{\pi^2}\frac{(\log b)^2}{b} + O\left(\frac{(\log(2b))^2}{b \log\log(3b)}\right). \tag{9}$$

In particular, $R_b \to 3$ *and*

$$\mathrm{Cov}_b(X_b^2, Y_b^2) \longrightarrow \frac{\pi^2}{3}.$$

Proof. The cubic law [6, Theorem 29] gives $E_b = b^3 + O(b^2(\log b)^2)$. The secondary-term theorem [7, Corollary 38] sharpens this to

$$E_b = b^3 - \frac{1}{\pi^2} b^2 (\log b)^2 + O\left(\frac{b^2(\log(2b))^2}{\log\log(3b)}\right).$$

Insert this formula into (6) and (7). Replacing $b-1$ by b in the lower-order denominators changes the answers by quantities absorbed in the stated errors. Finally,

$$\mathrm{Cov}_b(X_b^2, Y_b^2) = \mathbf{E}_b(X_b^2 Y_b^2) - \nu_b,$$

and $\nu_b \to \pi^2/6$. □

This is an averaged relation already supplied by collision energy. It does not identify r_b. The latter uses first magnitudes, while (6) uses their squares.

4. First Magnitude Moments

Put

$$a_b = \mathbf{E}_b X_b, \qquad c_b = \mathbf{E}_b Y_b, \qquad d_b = \mathbf{E}_b(X_b Y_b).$$

The exact marginal moments give

$$r_b = \frac{d_b - a_b c_b}{\sqrt{(1-a_b^2)(\nu_b - c_b^2)}}. \tag{10}$$

The task is to control these first moments, especially d_b. Character orthogonality expands the square moments into congruence counts. It does not remove the absolute values in $|T_b(\chi)|\,|B_b(\chi)|$.

The following finite comparison uses the same eighteen prime bases as the spectrum ledger. The nfield finite-character core [8] evaluates all 14,372 primitive odd characters. The Bernoulli sum supplies Y_b directly, without a prime cutoff. Rounded values at six of the bases are displayed below.

b	a_b	c_b	d_b	r_b	R_b
5	0.9177	1.1532	1.2566	1.0000	1.5000
7	0.8684	1.1317	1.2234	0.8440	1.8333
13	0.8308	1.1299	1.2055	0.8004	2.2381
29	0.8067	1.1307	1.1692	0.7203	2.5327
47	0.7935	1.1308	1.1502	0.6875	2.6720
71	0.7898	1.1308	1.1437	0.6753	2.7608

All eighteen bases satisfy the four moment targets $\mathbf{E}_b X_b^2$, $\mathbf{E}_b Y_b^2$, $\mathbf{E}_b X_b^2 Y_b^2$, and R_b within relative tolerance 2×10^{-12}. These numerical checks corroborate the identities proved above. The twisted identities in Proposition 8 are checked as complex equalities at the same bases, and the variance bound in Corollary 9 holds at all eighteen. The asymptotic theorem is established by the estimates below.

5. A Fractional-Moment Criterion

The short-sum first magnitudes become small even though their second moment stays equal to one. The following inequality converts that decay into a bound for the correlation. The required estimates are proved in Sections 6 and 7.

Proposition 7 (A sufficient decay criterion)**.** *Suppose, along odd prime bases, that*

$$H_b := \mathbf{E}_b X_b^{3/2} = o(1), \qquad \mathbf{E}_b Y_b^3 \le M, \qquad \mathrm{Var}_b(Y_b) \ge v_0 > 0 \tag{11}$$

for constants M, v_0 and all sufficiently large b. Then $r_b \to 0$. Whenever $H_b < 1$ and the two bounds hold,

$$|r_b| \le \frac{(M^{1/3} + \sqrt{\nu_b})H_b^{2/3}}{\sqrt{v_0(1 - H_b^{4/3})}}. \tag{12}$$

Proof. Hölder's inequality with exponents $3/2$ and 3 gives

$$d_b \le H_b^{2/3} M^{1/3}.$$

The monotonicity of probability-space moments gives $a_b \le H_b^{2/3}$, and Cauchy–Schwarz gives $c_b \le \sqrt{\nu_b}$. Therefore

$$|d_b - a_b c_b| \le (M^{1/3} + \sqrt{\nu_b})H_b^{2/3}.$$

Also $1 - a_b^2 \ge 1 - H_b^{4/3}$, while the assumed lower bound controls the other variance. Substitution into (10) proves (12). Its right side tends to zero under (11). ☐

6. Two L-Value Bounds

The variance can be bounded below without first evaluating the mean magnitude. Twisting the second moment by the character value at 2 gives the required separation. This identity is also the primitive-odd prime-square specialization of Louboutin's twisted quadratic moment [2, Theorem 1], recorded in [3, Section 1]. At twist two its cotangent correction vanishes. Subtracting the lifted odd characters gives the identity below; we supply the finite calculation.

Proposition 8 (The twist at two)**.** *For every odd prime $b \ge 5$,*

$$\mathbf{E}_b \chi(2) = 0, \qquad \mathbf{E}_b\big(Y_b^2 \chi(2)\big) = \frac{\nu_b}{2}. \tag{13}$$

Both are complex identities.

Proof. The first identity follows from Lemma 2 with $u = 2$ and $v = 1$. Neither $2 \equiv 1$ nor $2 \equiv -1$ holds modulo b or b^2.

For an odd modulus q, write $[2c]_q$ for the representative of $2c$ in $\{1, \dots, q-1\}$ and put

$$J(q) = \sum_{c=1}^{q-1} c\big(2[2c]_q - q\big).$$

Splitting at $(q-1)/2$ and summing the first two powers gives

$$J(q) = 4\sum_{c=1}^{q-1} c^2 - q\sum_{c=1}^{q-1} c - 2q \sum_{c=(q+1)/2}^{q-1} c$$
$$= \frac{q(q-1)(q-5)}{12}.$$

Odd-character orthogonality gives

$$\sum_{\substack{\psi \bmod q \\ \psi(-1)=-1}} \left| \frac{1}{q} \sum_{a\in\mathcal{U}_q} a\overline{\psi}(a) \right|^2 \psi(2)$$
$$= \frac{\varphi(q)}{2q^2} \sum_{c\in\mathcal{U}_q} c\big(2[2c]_q - q\big).$$

At modulus b^2, removing the multiples of b changes the inner sum to

$$J(b^2) - b^2 J(b) = \frac{b^2(b-1)(b^3-5)}{12}.$$

Thus the full odd-character twisted Bernoulli moment is

$$\frac{(b-1)^2(b^3-5)}{24b}.$$

The lifted odd characters contribute the corresponding moment modulo b. The Bernoulli sum is unchanged by lifting, as established in Proposition 3, and the value at 2 is unchanged as well. This contribution is

$$\frac{(b-1)^2(b-5)}{24b}.$$

Subtracting and dividing by N_b gives

$$\mathbf{E}_b\big(|B_b(\chi)|^2\chi(2)\big) = \frac{b^2-1}{12}.$$

Multiplication by π^2/b^2 proves the second identity. $\square$

Corollary 9 (A positive magnitude variance)**.** *For every odd prime $b \geq 5$,*

$$\mathrm{Var}_b(Y_b) \geq \frac{\nu_b}{16} \geq \frac{\pi^2}{100}. \tag{14}$$

Proof. Put $Z(\chi) = \Re\chi(2)$ and $c = \mathbf{E}_b Y_b$. Then $|Z| \leq 1$, $\mathbf{E}_b Z = 0$, and Proposition 8 gives

$$\frac{\nu_b}{2} = \mathbf{E}_b\big((Y_b - c)(Y_b + c)Z\big).$$

By Cauchy–Schwarz,

$$\begin{aligned}\frac{\nu_b}{2} &\le \sqrt{\operatorname{Var}_b(Y_b)}\sqrt{\mathbf{E}_b((Y_b+c)^2Z^2)}\\ &\le \sqrt{\operatorname{Var}_b(Y_b)}\sqrt{\nu_b+3c^2} \le 2\sqrt{\nu_b\operatorname{Var}_b(Y_b)}.\end{aligned}$$

Squaring proves the first bound. The second follows by inserting $b \ge 5$ in the expression for ν_b. □

Proposition 10 (A uniform fourth moment)**.** *Along odd prime bases $b \ge 5$,*

$$\mathbf{E}_b Y_b^4 \ll 1, \qquad \mathbf{E}_b Y_b^3 \ll 1.$$

The implied constants are absolute.

Proof. Put $q = b^2$. For every nonprincipal character modulo q, periodicity and the vanishing of a complete-period sum give

$$\left|\sum_{n\le t}\chi(n)\right| \le q.$$

Partial summation therefore gives, uniformly in such characters,

$$L(1,\chi) = A_q(\chi) + O(q^{-1}), \qquad A_q(\chi) = \sum_{n\le q^2}\frac{\chi(n)}{n}. \tag{15}$$

Indeed, the tail after an integer T has magnitude at most $2q/T$.

Write

$$A_q(\chi)^2 = \sum_{u\le q^4} c_u\chi(u),$$

where

$$0 \le c_u = \frac{\#\{n_1n_2 = u \mid n_1, n_2 \le q^2\}}{u} \le \frac{d(u)}{u}.$$

Here $d(u)$ is the divisor function. Averaging over all characters modulo q and using orthogonality yields

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}|A_q(\chi)|^4 = \sum_{\substack{u,v\le q^4\\ u\equiv v\ (q)\\ (uv,q)=1}} c_uc_v.$$

On the diagonal, the sum is at most $\sum_{u\ge1} d(u)^2/u^2 < \infty$. For the off-diagonal contribution, use $d(u) \ll_\varepsilon u^\varepsilon$ with $\varepsilon = 1/16$. This elementary bound follows from the prime-factorization formula for $d(u)$ by separating the finitely many small primes. Since $u, v \le q^4$, their two divisor factors have product $O(q^{1/2})$. Consequently

$$\sum_{\substack{u,v\le q^4\\ u\equiv v\ (q)\\ u\neq v}} c_u c_v \ll q^{1/2}\sum_{u\le q^4}\frac{1}{u}\sum_{1\le k\le q^3}\frac{1}{u+kq}$$
$$\ll q^{-1/2}(\log(2q))^2.$$

Thus the full-character fourth moment of A_q is bounded.

The primitive odd family has proportion $(b-1)/(2b)$ among all characters modulo b^2. Positivity gives

$$\mathbf{E}_b|A_q|^4 \le \frac{2b}{b-1}\frac{1}{\varphi(q)}\sum_{\chi \bmod q}|A_q(\chi)|^4 \ll 1.$$

Every character in this family is nonprincipal, so (15) proves the fourth-moment bound for Y_b. Hölder's inequality gives the third-moment bound. ☐

7. Short Sums at Prime-Square Modulus

Harper's theorem [1, Theorem 1] is stated for prime moduli. Its character-to-random comparison also works at modulus b^2 in the range needed here. The complete products being compared are units and remain below the modulus. We give the transfer explicitly.

Proposition 11 (The short-sum estimate)**.** *For odd prime $b \to \infty$,*

$$H_b = \mathbf{E}_b X_b^{3/2} \ll (\log\log b)^{-3/8}. \tag{16}$$

Proof. Set $m = b^2$, $x = b-1$, and denote the average over all characters modulo m by

$$\mathbf{E}_m^{\mathrm{all}} W = \frac{1}{\varphi(m)}\sum_{\chi \bmod m} W(\chi).$$

For units u, v with $1 \le u, v < m$, orthogonality gives

$$\mathbf{E}_m^{\mathrm{all}}\chi(u)\overline{\chi}(v) = \mathbf{1}_{u=v} = \mathbf{E} f(u)\overline{f(v)}, \tag{17}$$

where f is a Steinhaus random multiplicative function. Its values at primes are independent and uniform on the unit circle, and its other values are obtained multiplicatively.

The unit restriction is the only change in the exact comparison used in [1, Section 2.2, Lemma 1 and Propositions 1–2]. To verify it, retain the notation for the short prime cutoff P and use Harper's parameters with moment exponent 3/4. Choose P as in [1, Section 3.1], including the harmless rounding condition that $(\log P)^{0.01}$ is an integer, and put

$$\log P \asymp (\log x)^{1/6}, \qquad K = 2(\log P)^{1.02},$$
$$J = \lceil 1.2 \log\log P \rceil, \qquad \delta = (\log P)^{-1.3}.$$

Here K indexes the conditioning frequencies, and J is the range parameter in the smooth partition of unity. Write $\ell = \log P$ and $A = 2K + 1$ for the number of partition factors. The required comparison condition is

$$xP^{400(A/\delta)^2 \log(J\ell)} < m. \tag{18}$$

The logarithm of the factor multiplying x is

$$O\big(\ell^{5.64} \log \ell\big) = O\big((\log x)^{0.94} \log\log x\big) = o(\log x).$$

Since $m/x \asymp x$, condition (18) holds for all sufficiently large b.

To check the full products in Harper's Proposition 1, take its Taylor parameter

$$S = 100A \left\lfloor \delta^{-2} \log(J\ell) \right\rfloor.$$

Each partition function is approximated by a polynomial of degree $2S-1$. Its argument is a prime-and-prime-square sum, so each occurrence contributes an integer at most P^2. Expanding all A polynomials and the accompanying short-sum square gives character pairings $\chi(u)\overline{\chi}(v)$ with

$$u, v \leq xP^{2A(2S-1)} < xP^{4SA}, \qquad 4SA \leq 400(A/\delta)^2 \log(J\ell).$$

Thus (18) controls each entire product across all partition factors.

The same condition controls the Taylor remainders. A product of $j \leq A$ remainder majorants reduces by Hölder's inequality to even moments of order $2Sj$ of the prime-and-prime-square sums. In Harper's Lemma 1, the integers on each side of the corresponding pairing are bounded by

$$x(P^2)^{Sj} \leq xP^{2SA} < xP^{4SA} < m.$$

Every factor is either an integer $n \leq x = b - 1$ or a prime at most $P < b$. Hence all the products are units modulo b^2, and (17) applies term by term to both the polynomial comparisons and the remainder moments. The smooth-divisor bound used in that lemma is unchanged. If $\widetilde{d}(n)$ counts the P-smooth divisors of n, then

$$\sum_{n \leq x} \widetilde{d}(n) \leq x \prod_{p \leq P} (1 - p^{-1})^{-1} \ll x\ell.$$

Taking $x = 1$ in the comparison gives the partition-weight estimate of Harper's Proposition 2.

The removal of smooth integers in [1, Section 3.2] also uses only a second moment on integers at most x. Set $y = x^{1/\log\log x}$. These integers are units, and the contribution of the y-smooth terms to the $3/2$ moment is at most $\Psi(x, y)^{3/4}$, where $\Psi(x, y)$ counts y-smooth integers up to x. Harper's smooth-number estimate makes this $o(x^{3/4}(\log\log x)^{-3/8})$. Also $P < y < b$ eventually.

It remains to check that the comparison errors stay small after summing all conditioning boxes, as in [1, Section 3.3]. Put

$$\alpha = \frac{3}{4}, \qquad \eta = (J\ell)^{-A\delta^{-2}}, \qquad \mathcal{B} = (2J + 2)^A.$$

The polynomial comparison error is $O(x\eta)$ per box, and the partition-weight error is $O(\eta)$. The two accumulated losses are bounded by constant multiples of

$$x^\alpha(\mathcal{B}\eta)^\alpha \quad \text{and} \quad x^\alpha(\mathcal{B}\eta)^{1-\alpha}.$$

Both are $o(x^{3/4}(\log\log x)^{-3/8})$, since

$$\log(\mathcal{B}\eta) = A\{\log(2J + 2) - \delta^{-2}\log(J\ell)\} \le -\tfrac{1}{2}A\delta^{-2}\log(J\ell)$$

for all sufficiently large b. In the weighted conditioning inequalities the partition weights sum to one. Keeping their powers inside Hölder's inequality gives the displayed losses without requiring a positive lower bound for an individual weight.

Finally, the strongest precision restriction in [1, Sections 3.5–3.6] is

$$\delta \le \frac{1}{J\ell^{1.2}\sqrt{\log \ell}}.$$

It holds because $\delta J\ell^{1.2}\sqrt{\log \ell} = O(\ell^{-0.1}(\log \ell)^{3/2}) \to 0$. The weaker precision restriction follows, and $J \ge 1.2\log \ell$ by construction. The frequency range $|k| \le K = 2\ell^{1.02}$ covers the translated windows used in the random stage.

The comparisons yield Harper's equation (3.2), a random conditional expectation at exponent $\alpha = 3/4$, with negligible errors. Sections 3.4–3.6 then contain no character modulus. Their random Euler-product estimates apply without a further support restriction. Hence

$$\mathbf{E}_m^{\mathrm{all}}\left|\sum_{n\le x}\chi(n)\right|^{3/2} \ll \left(\frac{x}{1 + \frac{1}{4}\sqrt{\log\log P}}\right)^{3/4} \ll x^{3/4}(\log\log x)^{-3/8}.$$

Finally, positivity permits restriction to the primitive odd family,

$$\mathbf{E}_b\left|\sum_{n<b}\chi(n)\right|^{3/2} \le \frac{2b}{b-1}\mathbf{E}_m^{\mathrm{all}}\left|\sum_{n<b}\chi(n)\right|^{3/2}.$$

Divide by $b^{3/4}$. Complex conjugation does not change the magnitude of the short sum, so this proves (16) for the normalization of X_b. $\square$

8. Magnitude Correlation

Proof of Theorem 1. We have $H_b \ll (\log\log b)^{-3/8}$ by Proposition 11. Proposition 10 supplies an absolute upper bound for $\mathbf{E}_b Y_b^3$, and Corollary 9 supplies the lower bound $\mathrm{Var}_b(Y_b) \geq \pi^2/100$. Also

$$\mathbf{E}_b X_b \leq H_b^{2/3} \ll (\log\log b)^{-1/4},$$

so $\mathrm{Var}_b(X_b) = 1 - (\mathbf{E}_b X_b)^2 \to 1$. Both variances are therefore positive for all sufficiently large b. Proposition 7 now gives

$$|r_b| \ll H_b^{2/3} \ll (\log\log b)^{-1/4}.$$

$\square$

The limit $r_b \to 0$ does not make the mixed square moment equal to the product of its marginal moments. The square mass concentrates on a shrinking proportion of the characters.

Corollary 12 (Concentration of the square mass)**.** *For every fixed* $\varepsilon > 0$*,*

$$\frac{\#\{\chi \in \mathcal{P}_b \mid X_b(\chi) > \varepsilon\}}{N_b} \longrightarrow 0, \qquad \mathbf{E}_b\big(X_b^2 \mathbf{1}_{X_b > \varepsilon}\big) \longrightarrow 1.$$

At the same time, $R_b \to 3$ *and* $\mathrm{Cov}_b(X_b^2, Y_b^2) \to \pi^2/3$*.*

Proof. Markov's inequality bounds the displayed proportion by $\varepsilon^{-3/2} H_b$, which tends to zero. On $X_b \leq \varepsilon$,

$$\mathbf{E}_b\big(X_b^2 \mathbf{1}_{X_b \leq \varepsilon}\big) \leq \varepsilon^{1/2} H_b \longrightarrow 0.$$

Subtract from $\mathbf{E}_b X_b^2 = 1$. The final two limits are Corollary 6. $\square$

A vanishing proportion of the characters carries the short sum's square mass. The ordinary magnitude correlation tends to zero. The collision energy retains the excess in the squared weights, with limiting ratio three.

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