

THE UNIT SAMPLING DEFECT IN DIGIT-COLLISION ENERGY

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ABSTRACT. Carry counts from two-digit words with matching digits form a finite table. Its centered square mass is the digit-collision energy. We prove that its normalized energy is asymptotically one less than the exact continuous average, settling the sampling-limit conjecture left by the secondary-term calculation. For an odd prime p , put $N = p - 1$, let ψ be the centered sawtooth with value zero at integers, and set $B_N(x) = 2 \sum_{j \leq N} \psi(jx)$. Then

$$\int_0^1 B_N(x)^2 dx - \frac{1}{p^2} \sum_{a=0}^{p^2-1} B_N(a/p^2)^2 = 1 + O(p^{-\eta} \log^4(2p))$$

for an explicit absolute $\eta > 0$. The grid samples are exactly the centered carry readings in a different order. At the origin all clocks record zero, although their squared one-sided sum is N^2 . Its share N^2/p^2 tends to one. We show that the combined discrepancy from the interior resets tends to zero as both the profile and grid grow. The proof joins the two orientations of a Dedekind reciprocity descent into a complete interval with golden endpoints, then uses reciprocal-phase oscillation in its denominator. Replacing just the squared sample at the origin by N^2 therefore makes the finite and continuous averages agree asymptotically. In the original energy units, this identifies a quadratic sampling correction beneath the scale resolved by the cubic law and its secondary term.

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1. SAMPLING A GROWING CARRY TABLE

An integral ignores the value assigned at a single point. A finite table records it. When the table and the function grow together, that one value can leave a discrepancy that survives averaging. For the carry table studied here, the discrepancy tends to one. Its source is a common reset of all the clocks representing the table. The proof must also account for the growing family of resets between them.

A digit collision asks whether two digit positions agree. Here the positions are the two digits of a base- p word. In base three, the words 00, 11 and 22 occupy the integer positions zero, four and eight. Place them on a circle with nine positions. Multiplication by two moves their marks to zero, eight

and seven. A further step of two takes two marks across the join. Those crossings are the carries being counted.

For an odd prime base p , there are p matching-digit words on a circle of p^2 positions. For each multiplier a from zero through $p^2 - 1$, multiply their positions by a and count the crossings under a further step of a . The resulting list of counts is the *carry table*. Subtracting the proportional allowance a/p gives its centered readings. Their squares sum to the collision energy E_p .

The same table has a clock representation. Define the integer-zero sawtooth by

$$\psi(x) = \begin{cases} \{x\} - \frac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases} \quad B_N(x) = 2 \sum_{j=1}^N \psi(jx), \quad N = p-1,$$

where $\{x\}$ denotes the fractional part. Each component rises linearly and resets at multiples of $1/j$; this repeating motion is why we call it a clock. The clocks are added before the sum is squared, so their interactions are part of the energy.

The *sampling grid* consists of the p^2 points $0, 1/p^2, \dots, (p^2 - 1)/p^2$. Proposition 2.1 shows that the values of B_N on this grid are the centered carry readings in a different order. The continuous capacity and the finite average are therefore

$$\mathcal{J}(N) = \int_0^1 B_N(x)^2 dx, \quad \frac{E_p}{p^2} = \frac{1}{p^2} \sum_{a=0}^{p^2-1} B_N(a/p^2)^2.$$

Their difference $\mathcal{S}_p = \mathcal{J}(p-1) - E_p/p^2$ is the *sampling defect*. Sampling here uses every entry of the carry table.

The Secondary Term in Digit-Collision Energy [7] proved that, for odd prime bases,

$$E_p = p^3 - \frac{p^2}{\pi^2} (\log p)^2 + O\left(\frac{p^2 \log^2(2p)}{\log \log(3p)}\right).$$

That calculation bounded the sampling defect below the secondary scale and left its proposed limit $\mathcal{S}_p \rightarrow 1$ open. The limit resolves a finer question about the same energy, with its exact continuous capacity retained.

The origin explains the candidate one. Each doubled clock approaches -1 just to its right and $+1$ just to its left around the periodic join. Thus B_N^2 approaches N^2 from either side, while $B_N(0)^2 = 0$. Replacing that single squared sample by N^2 raises the finite average by N^2/p^2 , which tends to

one. To prove that this is the entire limiting difference, we must control the combined signed discrepancy from the interior. Both the number of clocks and the sampling grid grow with p .

The proof gives an explicit positive power saving. Its exponent combines the derivative-estimate saving $1/(2^{14} - 2)$ with a denominator cutoff at $p^{1/28}$, giving

$$\eta = \frac{1}{28(2^{14} - 2)} = \frac{1}{458696}.$$

These are conservative choices in the proof, explained at their points of use below. Their role is to establish convergence. The stronger square-root rate suggested by the numerical data is formulated in Conjecture 15.1.

Theorem 1.1 (Unit sampling defect). *For odd primes p , let $N = p - 1$ and*

$$\mathcal{S}_p = \int_0^1 B_N(x)^2 dx - \frac{1}{p^2} \sum_{a=0}^{p^2-1} B_N(a/p^2)^2.$$

With $\eta = 1/458696$, one has

$$\mathcal{S}_p = 1 + O(p^{-\eta} \log^4(2p)). \quad (1)$$

The implied constant is absolute. Equivalently, the original finite carry energy satisfies

$$E_p = p^2 [\mathcal{J}(p - 1) - 1] + O(p^{2-\eta} \log^4(2p)). \quad (2)$$

In the original energy units, the unit defect is a quadratic correction. It lies inside the larger remainder of the secondary-term expansion. The theorem determines this correction relative to the exact capacity $\mathcal{J}(p - 1)$; a finer expansion of that capacity is a separate question.

The sampling identity in Section 3 separates the origin from a signed sum of Dedekind terms. Reciprocity follows each term through division with remainders. An initial sequence of division steps is a *prefix*; its matrix records how the original pair is reconstructed from the smaller one. We stop when a remainder crosses a threshold and bound all the costs accumulated before that crossing. The terminal terms retain both possible orientations of the descent. Their signs allow them to be joined into a complete interval of primitive rational coordinates. Holding the numerator fixed then exposes reciprocal-phase oscillation in the denominator. This controls the terminal terms across all the original cutoffs and proves the limit.

The analytic ingredients are classical. We use Dedekind reciprocity and its cotangent representation [1], Weil's complete Kloosterman bound including its Ramanujan zero mode [6, Lemma 3.8], Erdős–Turán discrepancy [5, Theorem 8.1], and the classical derivative estimate stated in [4, equation (1)]. The golden interval belongs to the geometry of signed continued fractions [3]; its population is proved directly below. Throughout the descent, the signed source, its cutoff and its multiplicity remain those of the original carry table.

2. CLOCK REPRESENTATION OF THE CARRY TABLE

Fix an odd prime p throughout, and write

$$N = p - 1, \quad q = p^2, \quad \ell = \log(2p), \quad H_n = \sum_{j=1}^n \frac{1}{j}.$$

We write (j, k) for the greatest common divisor. A bar over a unit denotes its inverse modulo the modulus in use. The integers from zero through $q - 1$ represent all two-digit base- p words, including leading zeros. This finite residue system is the *two-digit carrier*. The word with both digits equal to d has value $pd + d$, so the equal-digit words occupy $G_p = \{(p+1)d : 0 \leq d < p\}$. For $0 \leq a < q$ put

$$C_p(a) = \sum_{n \in G_p} \left(\left\lfloor \frac{(n+1)a}{q} \right\rfloor - \left\lfloor \frac{na}{q} \right\rfloor \right), \quad F_p(a) = C_p(a) - \frac{a}{p},$$

and $E_p = \sum_{a=0}^{q-1} F_p(a)^2$. Each difference of floors is zero or one. It records whether the step from na/q to $(n+1)a/q$ crosses an integer boundary, counting arrival at the boundary as a crossing. This is the circular carry test described above.

A step of a crosses the join from exactly a of the q possible starting positions. The proportional allowance for the p selected marks is therefore $pa/q = a/p$. Subtracting it defines F_p without a random-digit assumption. The carry table is the list of $C_p(a)$; its centered table is the list of $F_p(a)$. These are the definitions used in [7].

Proposition 2.1 (Straightening the carry response). *Writing $[x]_q$ for the least residue, one has*

$$F_p([(1-p)A]_q) = B_N(A/q), \quad E_p = \sum_{A=0}^{q-1} B_N(A/q)^2.$$

Moreover,

$$\mathcal{J}(N) := \int_0^1 B_N(x)^2 dx = \frac{1}{3} \sum_{j,k \leq N} \frac{(j,k)^2}{jk}. \quad (3)$$

Proof. Expanding each floor into its argument minus its fractional part cancels the term a/p . Since $(1-p)(p+1)d \equiv d \pmod{q}$, the two fractional part sums at $a = [(1-p)A]_q$ have indices $0, \dots, p-1$ and $1-p, \dots, 0$. Their difference is

$$\sum_{j=1}^{p-1} (\{jA/q\} - \{-jA/q\}) = 2 \sum_{j=1}^{p-1} \psi(jA/q).$$

This includes integer arguments. The multiplier $1-p$ is a unit modulo q , proving the energy identity. At $a = pc$ the defining floors telescope to $C_p(pc) = c$, so the nonunit readings are indeed zero.

The L^2 Fourier coefficients of ψ are $-1/(2\pi in)$ for $n \neq 0$. Parseval, with the common frequencies of $\psi(jx)$ and $\psi(kx)$, gives

$$\int_0^1 \psi(jx)\psi(kx) dx = \frac{(j,k)^2}{12jk}.$$

Multiplication by four and summation prove (3). $\square$

Define the Dedekind sum for all integer h and positive k by

$$s(h, k) = \sum_{a=1}^{k-1} \psi(a/k)\psi(ha/k), \quad s(h, 1) = 0.$$

It is periodic and odd in h . On units it is invariant under inversion. We use the classical reciprocity identity

$$s(h, k) + s(k, h) = \mathcal{R}(h, k), \quad \mathcal{R}(h, k) = \frac{h^2 + k^2 + 1 - 3hk}{12hk}, \quad (h, k) = 1. \quad (4)$$

These formulas use the integer-zero convention throughout [1].

3. SEPARATING THE COMMON RESET

At an interior reset, the jump in the squared profile depends on the readings of the clocks that do not reset there. The signed source below records their combined contribution to the sampling defect.

Put $M_k = \lfloor N/k \rfloor$, $s_k = N - kM_k$ and

$$T_{k,s}(a) = 2 \sum_{j=1}^s \psi(ja/k), \quad \mathcal{A}_p = \sum_{k=2}^N M_k \sum_{a \in \mathcal{U}_k} T_{k,s_k}(a) \psi(qa/k),$$

where $\mathcal{U}_k$ is the group of units modulo k , represented in $1, \dots, k-1$.

Proposition 3.1 (Exact sampling identity). *The complete defect satisfies*

$$\mathcal{S}_p = \frac{N^2}{p^2} + \frac{4\mathcal{A}_p}{p^2} - \frac{N\mathbf{H}_N}{3p^3}. \quad (5)$$

Proof. Between jumps, B_N has slope $V = N(N+1)$. At a reduced fraction a/k , $2 \leq k \leq N$, its midpoint value is $T_{k,s_k}(a)$ and its jump is $-2M_k$. For $f = B_N^2$ this gives

$$[f]_{a/k} = -4M_k T_{k,s_k}(a), \quad [f']_{a/k} = -4VM_k.$$

At the origin the squared one-sided value is N^2 , the chosen sample is zero, and $[f']_0 = -4VN$. No nonzero jump lies on the q -grid, since $2 \leq k < p$ and $(k, q) = 1$.

Let $B_2(x) = \{x\}^2 - \{x\} + 1/6$. Distributional differentiation gives

$$f'' = 2V^2 + \sum_x [f']_x \delta_x + \sum_x [f]_x \delta'_x.$$

For nonzero frequencies this determines $\widehat{f}(n)$ by division by $(2\pi in)^2$. Summing the frequencies divisible by q , with symmetric Abel regularization for the first-derivative series, gives

$$\frac{1}{q} \sum_{A=0}^{q-1} f(A/q) - \int_0^1 f = -\frac{N^2}{q} + \frac{1}{q} \sum_x [f]_x \psi(qx) - \frac{1}{2q^2} \sum_x [f']_x B_2(qx).$$

Indeed $\sum_{j \neq 0} e^{-2\pi i j y} / (2\pi i j) = \psi(y)$ in this convention and the corresponding squared-denominator series is $-B_2(y)/2$. The finite jump sums commute with regularization. The origin correction replaces the regulated value N^2 by its sample zero.

The remaining quadratic term is evaluated completely. Expanding M_k as the number of multiples of k through N and reducing fractions gives

$$\frac{N}{6} + \sum_{k=2}^N M_k \sum_{a \in \mathcal{U}_k} B_2(qa/k) = \frac{N}{6} + \sum_{j=1}^N \sum_{a=1}^{j-1} B_2(qa/j) = \frac{\mathbf{H}_N}{6}.$$

The last equality follows because q permutes the residues modulo j and $\sum_{a=1}^{j-1} B_2(a/j) = (j^{-1} - 1)/6$. Substitution, using $V = pN$, proves (5). $\square$

The first term of (5) tends to one, and the harmonic term is $O(p^{-2} \log(2p))$. The sampling limit is therefore reduced to $\mathcal{A}_p = o(p^2)$. This is a bound on the combined signed contribution of the interior resets. The rest of the proof keeps that contribution together through its changing moduli and cutoffs.

For example, direct rational calculation at $p = 3, 5, 7$ gives

$$\mathcal{S}_p = \frac{11}{27}, \quad \frac{143}{225}, \quad \frac{347}{490}.$$

The asymptotic question is how the interior contribution behaves as the whole table grows.

For $L \geq 1$ and a nonzero residue $a = Q \bmod k$, define

$$\Delta_L(Q; k) = \mathbf{1}_{a \leq L \bmod k} - \mathbf{1}_{k-a \leq L \bmod k}, \quad \Delta_L(0; k) = 0. \quad (6)$$

Write $L_r = \lfloor N/r \rfloor$ and $c_{N,k}(Q) = \sum_{r \leq N/k} \Delta_{L_r}(Q; k)$. Thus Δ_L is odd, has absolute value at most one, and records both ends of the actual prefix.

Proposition 3.2 (The original count source). *One has*

$$\mathcal{A}_p = \sum_{k=2}^N \sum_{h \in \mathcal{U}_k} c_{N,k}(qh) s(h, k). \quad (7)$$

The source has mean zero, absolute value at most $\lfloor N/k \rfloor$, and cyclic variation at most $4\lfloor N/k \rfloor$.

Proof. Complete periods of $\psi(ja/k)$ sum to zero. We may therefore replace the terminal j -prefix in T_{k,s_k} by $1 \leq j \leq N$. Expanding M_k into multiples and combining reduced fractions yields

$$\mathcal{A}_p = 2 \sum_{l, j \leq N} s(j\bar{q}, l).$$

If $j = da, l = dk$ with $(a, k) = 1$, the sawtooth distribution identity $\sum_{b=0}^{d-1} \psi((x+b)/d) = \psi(x)$ gives $s(j\bar{q}, l) = s(a\bar{q}, k)$. Complete unit blocks in a vanish by oddness. At fixed d, k only the prefix ending at $L_d \bmod k$ survives. Change variables $a = qh \bmod k$, then pair h with $-h$ to turn twice this prefix into (6). The modulus-one term is zero. Each difference of interval indicators is odd, has zero mean and cyclic variation at most four. Summing gives the stated bounds. $\square$

There are two sawtooth conventions in the argument. Dedekind sums always use ψ . Expanding a literal floor uses $b(x) = \{x\} - 1/2$, including $b(n) = -1/2$ at integers. Direct division of Q and L by k proves, when $Q \not\equiv 0 \pmod{k}$,

$$\Delta_L(Q; k) = \frac{1}{k} - 2b(Q/k) + b((Q - L - 1)/k) + b((Q + L)/k). \quad (8)$$

The shift $-L - 1$ and the term $1/k$ will both be retained.

4. BOUNDS FOR WEIGHTED SOURCES

Three estimates will control different parts of the reciprocity descent. The first bounds the absolute mass of the Dedekind weights. The second uses oscillation along a reciprocal phase. The third handles a weight whose variation can be bounded while the original count source is retained. Here cyclic variation is the sum of absolute successive differences around the residue classes, and $\tau(k)$ counts the positive divisors of k .

Lemma 4.1 (Absolute Dedekind mass). *For every integer $k \geq 1$,*

$$\sum_{a \in \mathcal{U}_k} |s(a, k)| \leq \frac{k}{4} \mathbf{H}_k^2.$$

Consequently

$$\sum_{u \leq U} \sum_{\substack{1 \leq e < u/2 \\ (e, u) = 1}} |s(e, u)| \leq \frac{U(U+1)}{16} \mathbf{H}_U^2. \quad (9)$$

Proof. The cotangent representation [1, equations (3)–(4)] is

$$s(a, k) = \frac{1}{4k} \sum_{n=1}^{k-1} \cot(\pi n/k) \cot(\pi an/k), \quad (a, k) = 1.$$

The inequality $|\cot(\pi x)| \leq [2 \min(x, 1-x)]^{-1}$ gives $\sum_{n=1}^{k-1} |\cot(\pi n/k)| \leq k \mathbf{H}_k$. For fixed n , let $g = (n, k)$. Multiplication by n has at most g preimages of each nonzero residue, so

$$\sum_{a \in \mathcal{U}_k} |\cot(\pi an/k)| \leq g \sum_{j=1}^{k/g-1} |\cot(\pi j/(k/g))| \leq k \mathbf{H}_k.$$

The triangle inequality proves the first assertion, including composite k . Reflection halves this bound on $e < u/2$; its possible midpoint has zero weight. Sum $u/8$ over $u \leq U$ to obtain (9). $\square$

The order $k \log^2(2k)$ of this absolute mass is classical [2, Lemma 6]. The elementary proof above fixes a convenient bound for every modulus.

Lemma 4.2 (Reciprocal-phase discrepancy). *Let $|X| \geq 1$, $-1 \leq \beta \leq 2$, and $T \leq C_0 \sqrt{|X|}$. Uniformly over integer intervals $I \subset [2, T]$ and real γ ,*

$$\left| \sum_{t \in I} b\left(\gamma + \frac{X}{t + \beta}\right) \right| \ll_{C_0} |X|^{1/3} \log(2T).$$

Proof. On a dyadic interval of scale $Z \geq 2$, $t + \beta$ is comparable to Z . For frequency j , the second derivative has size $j|X|/Z^3$ and fixed sign. The classical second-derivative bound gives

$$\left| \sum_t e^{2\pi i j(\gamma + X/(t + \beta))} \right| \ll \sqrt{j|X|/Z} + Z^{3/2}/\sqrt{j|X|}$$

on any subinterval. Integrating the Erdős–Turán inequality gives

$$\left| \sum_{n=1}^m b(\theta_n) \right| \ll \frac{m}{J+1} + \sum_{j=1}^J \frac{1}{j} \left| \sum_{n=1}^m e^{2\pi i j \theta_n} \right|. \quad (10)$$

This formula includes integer phases, since integrating $m/2 - \#\{n : \{\theta_n\} < t\}$ over $0 < t < 1$ gives the left-hand sum before taking its absolute value. If $Z \leq 2|X|^{1/3}$ use the trivial bound. Otherwise choose $J = \lfloor Z/|X|^{1/3} \rfloor$ in (10). The result is $O(|X|^{1/3} + Z^{3/2}|X|^{-1/2}) = O_{C_0}(|X|^{1/3})$. Summing the dyadic pieces proves the claim. Short terminal intervals obey the same estimates. $\square$

Lemma 4.3 (Completion against the count source). *Let f be a real function on residues modulo k , with $f(0) = 0$ and cyclic variation at most V_k . Then*

$$\left| \sum_{c \in \mathcal{U}_k} f(c) c_{N,k}(q\bar{c}) \right| \ll \frac{NV_k}{k} \tau(k) \left(\sqrt{k} \log^2(2k) + \log(2k) \right). \quad (11)$$

Proof. Use normalized coefficients $\widehat{f}(m) = k^{-1} \sum_c f(c) e^{-2\pi i mc/k}$ with centered frequency representatives. Summation by parts, and the fact that a cyclic path returns to zero, give

$$\begin{aligned} |\widehat{f}(0)| &\leq V_k/2, & |\widehat{f}(m)| &\leq \frac{V_k}{4|m|} & (m \neq 0), \\ \widehat{c}(0) &= 0, & |\widehat{c}(n)| &\leq \frac{N}{k|n|} & (n \neq 0). \end{aligned}$$

Fourier inversion expresses the sum exactly as

$$\sum_m \sum_{n \neq 0} \widehat{f}(m) \widehat{c}(n) K(m, nq; k),$$

where

$$K(m, n; k) = \sum_{x \in \mathcal{U}_k} e^{2\pi i(mx + n\bar{x})/k}.$$

There is no extra factor k . Weil's bound and the Ramanujan bound are

$$|K(m, n; k)| \ll \tau(k) \sqrt{k} \sqrt{(m, n, k)}, \quad |K(0, n; k)| \leq (n, k).$$

Since $(q, k) = 1$, multiplication by q does not change the gcd factors. For nonzero m, n ,

$$\sum_{m, n \neq 0} \frac{\sqrt{(m, n, k)}}{|mn|} \ll \log^2(2k) \sum_{d|k} d^{-3/2} \ll \log^2(2k).$$

For the retained mode $m = 0$, $\sum_{n \neq 0} (n, k)/|n| \ll \tau(k) \log(2k)$ by expansion over common divisors. Substitution proves (11). $\square$

5. THE FIRST RECIPROCITY STEP

Choose a threshold H for the smaller member of a Dedekind pair. The pairs already below it admit a direct reciprocal-phase estimate. For the others, reciprocity separates a rational correction from a new Dedekind sum at a smaller modulus.

Inversion and reflection in (7) give

$$\mathcal{A}_p = 2 \sum_{k=2}^N \sum_{\substack{0 < c < k/2 \\ (c, k)=1}} s(c, k) c_{N, k}(q\bar{c}). \quad (12)$$

The exceptional unit at $k = 2$ has zero Dedekind sum. For $H \geq 1$, let $\mathcal{L}(H)$ be the part with $c \leq H$. On $c > H$, apply (4) to write

$$\mathcal{A}_p = \mathcal{L}(H) + \mathcal{E}(H) + \mathcal{B}(H), \quad (13)$$

where $\mathcal{E}$ uses $2\mathcal{R}(c, k) c_{N, k}(q\bar{c})$ and $\mathcal{B}$ uses $-2s(k, c) c_{N, k}(q\bar{c})$ on the same domain.

Proposition 5.1 (First-stage bounds). *Uniformly for integers $1 \leq H \leq N$,*

$$|\mathcal{L}(H)| \ll p^{5/3} H^{1/3} \ell^3, \quad |\mathcal{E}(H)| \ll p^{5/2} H^{-1} \ell^3.$$

Proof. For the first estimate fix c and write $k = ct + a$ with $a \in \mathcal{U}_c$. When $c = 1$, use $a = 0$ and its single residue class. Put $f = \bar{a} \bmod c$, with $f = 0$ at $c = 1$. Then

$$\frac{\bar{c}}{k} \equiv -\frac{f}{c} + \frac{1}{ck} \pmod{1}, \quad \frac{q\bar{c} + v}{k} \equiv -\frac{qf}{c} + \frac{q/c^2 + v/c}{t + a/c} \pmod{1}.$$

For each source cutoff $L = L_r$, the shifts are $v = 0, -L - 1, L$. An active family has $c < k/2 < p/2$. Thus its three amplitudes are comparable to p^2/c^2 , and $t \leq L/c \ll \sqrt{|X|}$. Conditions $2c < k \leq L$ give a single interval. Lemma 4.2, applied to (8), bounds every partial source sum by $O((p/c)^{2/3}\ell)$. The term $1/k$ costs $O(\ell/c)$ and is absorbed.

The actual weight is

$$s(c, ct + a) = \frac{t + a/c}{12} - s(a, c) - \frac{1}{4} + \frac{1 + c^{-2}}{12(t + a/c)}. \quad (14)$$

On $t + a/c > 2$ its endpoint size plus total variation is $O(L/c + 1 + |s(a, c)|)$. Partial summation and then summation over $r \leq N/(2c)$ give, for fixed c, a ,

$$\ll \frac{p^{5/3}}{c^{5/3}} (\ell^2 + (1 + |s(a, c)|)\ell).$$

Lemma 4.1 bounds the sum of these baselines over a by $O(c \log^2(2c))$. Summing $c^{-2/3}$ through H proves the first claim. This calculation includes the $c = 1$ families directly.

For the second estimate extend $\mathcal{R}(c, k)$ by zero outside $H < c < k/2$. It is decreasing on that interval, its absolute value is at most $k/(4H)$, and its cyclic variation is at most k/H . Apply Lemma 4.3 with $V_k = k/H$ and include the factor two. The elementary bound $\sum_{k \leq N} \tau(k) \leq N H_N$ finishes the sum. $\square$

6. CANCELLATION ACROSS MODULI

For a state in $\mathcal{B}(H)$, one has $H < c < k/2$ and $(c, k) = 1$. The states with $c = 1$ are already included in $\mathcal{L}(H)$. After removing the zero-weight tie described below, write the strict nearest-remainder step

$$k = mc + \varepsilon d, \quad 0 < d < c/2, \quad m \geq 2 \ (\varepsilon = 1), \quad m \geq 3 \ (\varepsilon = -1).$$

The only coprime tie has $(c, d) = (2, 1)$ and weight $s(k, 2) = 0$; its contribution is assigned zero. Let $\mathcal{C}(H, D)$ be the portion of $\mathcal{B}(H)$ with $d \leq D$, and let $\mathcal{Z}(H, D)$ be its complement. This partitions the original source states without changing their weights.

Proposition 6.1 (Cross-modulus bound). *For $1 \leq H, D \leq N$,*

$$|\mathcal{C}(H, D)| \ll p^{5/3} D^{1/3} \ell^3 + p D \ell^4. \quad (15)$$

Proof. Fix m, d, ε and write $c = dt + a$, $a \in \mathcal{U}_d$. At $d = 1$ use $a = 0$. Then $s(k, c) = \varepsilon s(d, c)$ and

$$\frac{\bar{c}}{k} \equiv \frac{\varepsilon \bar{a}}{d} - \frac{\varepsilon m}{dk} \pmod{1}.$$

For $v = 0, -L - 1, L$, put

$$\beta = \frac{a}{d} + \frac{\varepsilon}{m}, \quad X_v = -\frac{\varepsilon p^2}{d^2} + \frac{v}{md}.$$

The shifted phase is $\varepsilon q \bar{a}/d + X_v/(t + \beta)$ modulo one. Here $-1/3 \leq \beta \leq 3/2$ and $\frac{7}{8}p^2/d^2 \leq |X_v| \leq \frac{9}{8}p^2/d^2$, since $d < c/2 < k/4 < p/4$ and $|v| \leq p$. Also $t \leq L/(md) + 1/3 < \sqrt{2|X_v|}$. The conditions $c > H$, $c > 2d$ and $k \leq L$ give one interval. Thus every partial source sum has size $O((p/d)^{2/3} \ell)$, including the $1/k$ term.

Split $s(d, c) = \mathcal{R}(d, c) - s(a, d)$. The size plus variation of the first weight is $O(L/(md))$ on this interval. For fixed m, d, a, ε , partial summation and the source sum give $O(p^{5/3} m^{-1} d^{-5/3} \ell^2)$. Sum over at most d residues, both signs, $m \leq N$ and $d \leq D$. The result is $O(p^{5/3} D^{1/3} \ell^3)$.

The next baseline is paid separately, at its actual modulus d . Since $k > mc/2$, $|c_{N,k}| \leq N/k$, and $\sum_{c \equiv a \pmod{d}, c > 2d}^N 1/c \leq H_N/d$, its absolute contribution at fixed m, d, ε is at most $N H_N H_d^2/m$. Lemma 4.1 supplies the sum over a . Both orientations and all m, d therefore cost at most $2NDH_N^2 H_D^2$, proving (15). $\square$

7. DESCENT TO THE THRESHOLD

The preceding step controls pairs that reach a small remainder immediately. For the other pairs we repeat division, retaining both the correction at each step and the Dedekind sum still to be resolved.

We now take $D = H$ and retain $\mathcal{Z}(H, H)$. Start its original pair at $u_0 = k, u_1 = c$ and descend by

$$u_{j-1} = m_j u_j + \varepsilon_j u_{j+1}, \quad 0 < u_{j+1} < u_j/2.$$

Perform a step only when $u_{j+1} > H$, stopping before the first step crossing to $u_{j+1} \leq H$. Let $J \geq 1$ be the number performed and $\delta_0 = 1, \delta_j = -\varepsilon_j \delta_{j-1}$.

Reciprocity gives

$$\begin{aligned} s(k, c) &= f_{k,H}(c) + t_{k,H}(c), \\ f_{k,H}(c) &= - \sum_{j=1}^J \delta_j \mathcal{R}(u_{j+1}, u_j), \\ t_{k,H}(c) &= \delta_J s(u_J, u_{J+1}). \end{aligned} \tag{16}$$

The corresponding split of $\mathcal{Z}$ is denoted $\mathcal{Z}(H, H) = \mathcal{K}(H) + \mathcal{T}(H)$, with the original coefficient $-2c_{N,k}(q\bar{c})$ attached to both terms.

A finite nonempty word $((m_1, \varepsilon_1), \dots, (m_s, \varepsilon_s))$ is *admissible* when each $\varepsilon_i \in \{-1, 1\}$ and each integer m_i satisfies $m_i \geq 2$ if $\varepsilon_i = 1$ and $m_i \geq 3$ if $\varepsilon_i = -1$. Matrix products are ordered from the first digit on the left to the last on the right. These are exactly the digit restrictions of the strict nearest-remainder descent above. The matrix of a prefix carries the terminal remainder pair back to its original pair. Its determinant, equal to $+1$ or -1 , records the orientation of that reconstruction.

Lemma 7.1 (Prefix population and stopped variation). *For an admissible nonempty word put*

$$P = \prod_i \begin{pmatrix} m_i & \varepsilon_i \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & E \end{pmatrix}, \quad \det P = \delta.$$

Then $A \geq 2$, $|B| < A$, $0 < C \leq A/2$, and $(A, C) = 1$. Distinct words give distinct matrices, and there are at most $4A$ matrices with a given A . The stopped cost has an extension to all residues satisfying

$$f_{k,H}(0) = 0, \quad \text{TV}(f_{k,H}) \leq 8(k/H)^2. \tag{17}$$

Proof. Appending a digit changes the top row to $(Am + B, \varepsilon A)$, so A strictly increases and $|B| < A$. Apply the word backwards to a ratio $e/u \in (0, 1/2)$. Every step produces a ratio in $(0, 1/2)$, proving $0 < C \leq A/2$ by taking $e/u \downarrow 0$. The determinant gives $(A, C) = 1$. Given A, C, δ , the congruence $BC \equiv -\delta \pmod{A}$ leaves at most two choices of B in $(-A, A)$, and then fixes E . This bounds the count by $4A$. The nearest-quotient procedure is unique on each word's nonempty interval. Two words with the same matrix coincide until one ends; strict growth of A excludes a proper extension with the same matrix.

At a performed cost (u, e) , $e > H$ and $u > 2e$, while

$$k = Au + Be, \quad c = Cu + Ee, \quad e = \delta(Ac - Ck), \quad u = (k - Be)/A.$$

Its support is one interval in c . Reversing the word shows that earlier steps are all above H and that the original ratio lies in $(0, 1/2)$. In particular $k > A(u - e) > AH$ and $u < 2k/A$. For this prefix, extend the cost $-\delta\mathcal{R}(e, u)$ by zero off that interval. Its absolute value is at most $k/(2AH)$. To see monotonicity in e , put $v = u/e = k/(Ae) - B/A > 2$. Both $v + 1/v$ and $1/(eu)$ decrease, the latter because $(eu)' = (k - 2Be)/A > 0$. Thus $\mathcal{R}(e, u)$ decreases. Its interior variation and its two endpoint jumps total at most $2k/(AH)$. Sum over all prefixes with $A < k/H$ to get

$$\text{TV}(f_{k,H}) \leq \sum_{2 \leq A < k/H} 4A \frac{2k}{AH} \leq 8(k/H)^2.$$

On unit integers this extension is exactly (16). On nonunits it is defined by the same interval sum, which is all Fourier completion requires. Strict endpoints are retained. $\square$

Proposition 7.2 (Complete stopped-cost bound). *Uniformly for $1 \leq H \leq N$,*

$$|\mathcal{K}(H)| \ll p^{7/2} H^{-2} \ell^3.$$

Consequently, at $H = \lfloor p^{11/14} \rfloor$,

$$\mathcal{S}_p = 1 + \frac{4\mathcal{T}(H)}{p^2} + O(p^{-1/14} \ell^3). \quad (18)$$

Proof. Apply Lemma 4.3 to (17), retaining the factor -2 . Its bound at modulus k is

$$\ll \frac{N}{H^2} \tau(k) (k^{3/2} \log^2(2k) + k \log(2k)).$$

Sum using $\sum_{k \leq N} \tau(k) \leq N\mathbf{H}_N$. The exact full-source identity is

$$\mathcal{A}_p = \mathcal{L}(H) + \mathcal{E}(H) + \mathcal{C}(H, H) + \mathcal{K}(H) + \mathcal{T}(H). \quad (19)$$

Propositions 5.1 and 6.1 control the other terms. At the stated threshold their largest power is $p^{27/14}$. The term $pH\ell^4$ has smaller power $25/14$, so its extra logarithm is absorbed. Insert these bounds into (5). No source-index truncation is made. $\square$

The threshold $p^{11/14}$ balances the growing cost $p^{5/3} H^{1/3}$ against the decreasing cost $p^{7/2} H^{-2}$. Both have size $p^{27/14}$ at that scale. After division by p^2 , all the accumulated corrections therefore vanish. It remains to obtain a saving for the terminal sum $\mathcal{T}(H)$.

8. THE GOLDEN PREFIX INTERVAL

To estimate the terminal sum, we first need its full population. The matrices of the stopped paths retain both orientations. Their top rows will turn the varying path lengths into a single interval of rational coordinates.

Include the first crossing step in P . Its word has length at least two. The terminal domain for source index r is

$$\begin{aligned} u > H, \quad 1 \leq e \leq H, \quad 2e < u, \quad (u, e) = 1, \\ k = Au + Be \leq L_r, \quad c = Cu + Ee. \end{aligned} \quad (20)$$

The coprime tie $(u, e) = (2, 1)$ has zero weight and is assigned zero contribution before imposing the strict inequality $2e < u$. All preceding remainders exceed u , so this is precisely a first crossing. For the sign, write the crossing step as $u_J = mu + \varepsilon e$, where $u = u_{J+1}$. Periodicity and oddness give the residual $\delta_J s(u_J, u) = \delta_J \varepsilon s(e, u)$. The original coefficient -2 and the update $\delta = \delta_{J+1} = -\varepsilon \delta_J$ therefore give

$$-2\delta_J \varepsilon s(e, u) = 2\delta s(e, u).$$

Consequently

$$F_r(P) = 2\delta \sum_{(u,e) \text{ in (20)}} s(e, u) \Delta_{L_r}(q\bar{c}; k), \quad \mathcal{T}(H) = \sum_{r=1}^N \sum_P F_r(P). \quad (21)$$

Every state retains its original source coefficient and multiplicity.

Put

$$\alpha = \frac{\sqrt{5} - 1}{2}, \quad \beta = 1 - \alpha, \quad I = (-\beta, \alpha).$$

Theorem 8.1 (Complete prefix interval). *Admissible words of length at least two are in bijection with*

$$A \geq 2, \quad (A, B) = 1, \quad |B| \geq 2, \quad B/A \in I. \quad (22)$$

The top row determines the second row and the orientation uniquely. Both endpoints of I are limits of admissible words.

Proof. Appending a digit sends $z = B/A$ to $\varepsilon/(m + z)$. The identities $\beta = 1/(3 - \beta)$, $\alpha = 1/(2 - \beta)$ and $\alpha + \beta = 1$ show that all admissible maps preserve I . Starting at $(1, 0)$ gives primitive top rows. At length at least two, $|B|$ is the preceding denominator and is at least two.

Conversely, for a nonzero reduced $B/A \in I$, set

$$A' = |B|, \quad \varepsilon = \operatorname{sgn} B, \quad m = \lfloor A/|B| + \beta \rfloor, \quad B' = A - m|B|.$$

The interval has length one with irrational endpoints, so m is the unique integer with $B'/A' \in I$. Its endpoint inequalities imply $m \geq 2$ for $B > 0$ and $m \geq 3$ for $B < 0$. The denominator decreases strictly, and the pair remains primitive. The process ends at $(1, 0)$. Reversal reconstructs a unique word. The pairs $|B| = 1$ are exactly its one-step words.

For an explicit second row, let b_0 be the inverse of B modulo A . At length at least two, $A \geq 5$ and $0 < C < A/2$. The determinant gives

$$C = \min(b_0, A - b_0), \quad \delta = \begin{cases} -1, & b_0 < A/2, \\ 1, & b_0 > A/2, \end{cases} \quad E = (\delta + BC)/A.$$

There is no unit midpoint tie. Finally, if F_j are the Fibonacci numbers, the powers of $\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$ have $B/A = -F_{2j}/F_{2j+2} \rightarrow -\beta$. Appending $(2, 1)$ gives $B/A = F_{2j+2}/F_{2j+3} \rightarrow \alpha$. $\square$

The inverse ratio map is $z \mapsto |z|^{-1} - \lfloor |z|^{-1} + 1 - \alpha \rfloor$, an α -continued-fraction map in the classical family [3]. The theorem concerns its finite prefix population. This explicit description lets us sum over every admissible path while retaining its original source weight.

9. SMALL PREFIX BLOCKS

We group the terminal prefixes by the size of their denominator A . For small denominators, reciprocal-phase oscillation in the terminal pair gives a sufficient bound. Larger denominators will require the joined population from the golden interval.

For dyadic K let $\mathcal{B}_r(K; H) = \sum_{K \leq A < 2K} F_r(P)$, including both orientations. For an active matrix, $k > A(u - e) > Au/2$, so $AH < 2L_r$ and $u < 2L_r/K$. Lemma 4.1 therefore gives the block's terminal absolute mass

$$W_{r,K} := \sum_{\text{active } (u,e)} |s(e, u)| \ll (L_r/K)^2 \ell^2. \quad (23)$$

Proposition 9.1 (Small-block estimate). *For every active r, K, H ,*

$$|\mathcal{B}_r(K; H)| \ll p^{2/3} L_r H^{1/3} K \ell^3. \quad (24)$$

Proof. Fix a matrix and write $u = et + a$, with $a \in \mathcal{U}_e$ and the convention $a = 0$ at $e = 1$. Set $f = \bar{a} \bmod e$, with $f = 0$ at $e = 1$. The determinant

identity gives

$$\frac{q\bar{c} + v}{k} \equiv -\frac{\delta q f}{e} + \frac{\delta q/e^2 + v/(Ae)}{t + a/e + B/A} \pmod{1}. \quad (25)$$

For example, $h = \delta(A(1 - fu)/e - Bf)$ is integral and satisfies $ch \equiv 1 \pmod{k}$, proving the formula. The shifts are again $v = 0, -L_r - 1, L_r$. Because $e < k/A < p/A$, $|v|e/(Aq) < 1/A^2 \leq 1/4$. Thus the amplitudes are comparable to p^2/e^2 . Also $-1 < a/e + B/A < 2$ and $t \ll p/e$. The terminal inequalities and the cutoff leave one interval in t . Lemma 4.2 bounds every partial source sum by $O((p/e)^{2/3}\ell)$, including the literal $1/k$ terms.

Formula (14), with $c = e$, bounds the endpoint size plus variation of the full $s(e, et + a)$ by $O(L_r/(Ae) + 1 + |s(a, e)|)$. Partial summation and Lemma 4.1 give

$$|F_r(P)| \ll \left(\frac{p^{2/3} L_r H^{1/3}}{A} + p^{2/3} H^{4/3} \right) \ell^3.$$

The two contributions use, respectively, the estimates

$$\sum_{e \leq H} e^{-2/3} \ll H^{1/3}, \quad \sum_{e \leq H} e^{1/3} \log^2(2e) \ll H^{4/3} \log^2(2H).$$

Since $AH < 2L_r$, the second term is absorbed by the first. There are at most $4A$ matrices at each A . Sum over $K \leq A < 2K$ to obtain (24). $\square$

10. FOLDING OPPOSITE ORIENTATIONS

The small-block bound grows with K . For large blocks we instead sum over the prefix population before estimating its size. The following identity transfers the orientation sign into the odd count source.

Proposition 10.1 (Orientation fold). *Fix (u, e) in (20). Put*

$$f = \bar{u} \bmod e, \quad v_0 = (1 - fu)/e, \quad h = Av_0 - Bf = (A - fk)/e,$$

using $f = 0$ when $e = 1$. Then

$$\delta\Delta_L(q\bar{c}; k) = \Delta_L(qh; k). \quad (26)$$

Consequently

$$\mathcal{B}_r(K; H) = 2 \sum_{\substack{u > H, \ 1 \leq e \leq H \\ 2e < u, \ (u, e) = 1}} s(e, u) \tilde{G}_{r, K}(u, e), \quad (27)$$

where

$$\tilde{G}_{r,K}(u, e) = \sum_{\substack{K \leq A < 2K, \ |B| \geq 2, \ B/A \in I \\ (A,B)=1, \ Au+Be \leq L_r}} \Delta_{L_r}(q(Av_0 - Bf); Au + Be). \quad (28)$$

Proof. From $Ac - Ck = \delta e$ one obtains $ch = \delta + k(C - fc)/e$. The quotient is integral since $c \equiv Cu \pmod{e}$ and $fu \equiv 1 \pmod{e}$. Thus $\bar{c} \equiv \delta h \pmod{k}$, and oddness of Δ_L proves (26). Substitute in (21) and apply Theorem 8.1. Both orientations together give precisely the primitive pairs in (28). $\square$

For one orientation alone the inverse of B modulo A selects only half the population. Equation (26) removes that restriction from the joined source. Each original contribution keeps its value, and the complete population now admits the interval summation used next.

11. OSCILLATION IN THE PREFIX DENOMINATOR

Write $L = L_r$. Holding $B \neq 0$ fixed, the restrictions in (28), apart from coprimality, are a single interval

$$K \leq A < 2K, \quad A > \begin{cases} B/\alpha, & B > 0, \\ -B/\beta, & B < 0, \end{cases} \quad A \leq (L - Be)/u. \quad (29)$$

Globally $-2\beta K < B < 2\alpha K$. Expand coprimality by

$$\mathbf{1}_{(A,B)=1} = \sum_{d|A, d|B} \mu(d), \quad A = da. \quad (30)$$

Here μ is the Möbius function. On each resulting progression the literal count source becomes a sum of reciprocal phases. Its endpoint shifts must be retained even when coprimality has been removed.

Lemma 11.1 (Nonprimitive extension and endpoint phases). *The literal identity (8) remains valid on every pair introduced by (30). For $v = 0, -L - 1, L$, put*

$$\gamma = -qf/e + q/(eu), \quad \zeta = eB/u, \quad X_v = qB/u^2 - v/u.$$

Then, as a real identity,

$$\frac{qh + v}{k} = \gamma - \frac{X_v}{A + \zeta}. \quad (31)$$

For every nonempty block,

$$u < 2L/K \leq 2p/K, \quad K \geq 4, \quad (32)$$

$$\frac{3}{4} \frac{p^2 |B|}{u^2} \leq |X_v| \leq \frac{5}{4} \frac{p^2 |B|}{u^2}, \quad \alpha K < A + \zeta < (2 + \alpha)K \quad (K \leq A \leq 2K). \quad (33)$$

Proof. The map

$$\begin{pmatrix} k \\ h \end{pmatrix} = \begin{pmatrix} u & e \\ v_0 & -f \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

has determinant -1 , so $(k, h) = (A, B)$. Since $|B| < A$ and $u > 2e$, $k > A(u - e) > A$, whence $(k, h) < k$. Since $k \leq L < p$, it follows that $qh \not\equiv 0 \pmod{k}$ even for nonprimitive pairs.

Substitute $h = (A - fk)/e$ and $k = u(A + \zeta)$ to get (31). The inequality $k > Au/2$ gives (32); the first possible top-row denominator is $A = 5$. Since $|v| \leq p$ and $|B| \geq 2$, $|vu/(qB)| < 2/(K|B|) \leq 1/4$, giving the amplitude bounds. Finally $e/u < 1/2$ and the global range of B give $-\beta K < \zeta < \alpha K$, proving the denominator bounds on the entire real interval. $\square$

The phase amplitudes range too widely for one fixed derivative order to give a saving everywhere. Orders two through fourteen cover the window below. The weakest of their savings is $\kappa = (2^{14} - 2)^{-1}$, which is the derivative factor in the exponent announced in Section 1.

Lemma 11.2 (A fixed derivative window). *Let $M \geq 2$. Suppose a real phase on an interval of length at most M has derivatives of constant sign satisfying $|g^{(j)}| \asymp_j TM^{-j}$ for $2 \leq j \leq 14$, with fixed comparison constants. If $M^{1/2} \leq T \leq M^{13}$, then every integer subinterval satisfies*

$$\left| \sum_n e^{2\pi i g(n)} \right| \ll M^{1-\kappa}, \quad \kappa = \frac{1}{16382}.$$

Proof. The classical j th derivative bound [4, equation (1)] is

$$\left| \sum_n e^{2\pi i g(n)} \right| \ll_j m \lambda^{a_j} + m^{1-b_j} \lambda^{-a_j}, \quad a_j = (2^j - 2)^{-1}, \quad b_j = 2^{2-j},$$

on length m , when the derivative has fixed sign and magnitude comparable to λ . For $T \leq M$, the second derivative gives $O(\sqrt{T} + M/\sqrt{T}) = O(M^{3/4})$. Otherwise choose $3 \leq j \leq 14$ with $M^{j-2} \leq T \leq M^{j-1}$. Then $\lambda \asymp TM^{-j}$ lies between constant multiples of M^{-2} and M^{-1} . The two terms are bounded

by M^{1-a_j} and $M^{1-b_j+2a_j}$. Since $b_j - 2a_j \geq a_j \geq \kappa$, both give the required saving. For a shorter interval retain the same derivative scale and replace its actual length by $O(M)$ in the nonnegative powers above. Empty intervals and intervals with at most one integer cost $O(1)$. $\square$

12. POWER SAVING FOR LARGE BLOCKS

At $H = \lfloor p^{11/14} \rfloor$, split the prefix denominators at $K_0 = p^{1/28}$. The small-block estimate then gives a total cost of order $p^{55/28} \ell^4$, below p^2 . Above that cutoff, the reciprocal phases fit the derivative window of Lemma 11.2. The resulting factor satisfies $K^{-\kappa} \leq p^{-\kappa/28}$ in that range, giving the exponent $\eta = \kappa/28$.

Theorem 12.1 (Joined source estimate). *Let $H = \lfloor p^{11/14} \rfloor$. Uniformly in every active source index and terminal pair,*

$$|\tilde{G}_{r,K}(u, e)| \ll K^{2-\kappa} \log(2K), \quad K > p^{1/28}, \quad \kappa = 1/16382. \quad (34)$$

Proof. Set $D = J = \lfloor K^{1/100} \rfloor$. First retain the divisors $d \leq D$ in (30), and put $M = K/d \geq K^{99/100}$. For $1 \leq j \leq J$ the phase on the progression $A = da$ is

$$g(a) = j\gamma - \frac{jX_v}{da + \zeta}, \quad |g^{(h)}(a)| \asymp_h TM^{-h}, \quad T = j|X_v|/K.$$

The comparison constants in Lemma 11.1 are uniform in all source and terminal parameters. Its lower bound gives $T \geq 3K/8$. Its upper bound, $u > H \geq p^{11/14}/2$ and $|B| < 2K$, gives

$$T \leq 10jp^{3/7} < 10K^{12+1/100}.$$

For $K \geq 16$, these imply $M^{1/2} \leq T \leq M^{13}$. For the upper inequality use $13(99/100) = 12.87$ and $10 \leq 16^{0.86}$. For the lower one use $M \leq K$. The fixed dyadic cases $K = 4, 8$ obey (34) after enlarging the absolute constant, by counting their $O(K^2)$ pairs.

Apply Lemma 11.2 and (10) on the actual interval (29). For each fixed $B, d \leq D$,

$$\left| \sum_a \Delta_L(qh; k) \right| \ll \frac{M}{J} + M^{1-\kappa} \log(2J) + \frac{1}{du}. \quad (35)$$

The last term pays all the $1/k$ terms, because $k \asymp Ku$ and there are $O(K/d)$ integers in this interval. Integer hits of the shifted phases are included by the literal convention in (10).

For a given d there are $O(K/d)$ nonzero multiples of d in the global B range. Summing (35) and using $\sum d^{-2} < \infty$, $\sum d^{\kappa-2} < \infty$ gives

$$\ll K^2/J + K^{2-\kappa} \log(2K) + K/u.$$

For $d > D$ use $|\Delta_L| \leq 1$. Their total cost is

$$\ll \sum_{D < d < 2K} \frac{K}{d} \left(\frac{K}{d} + 1 \right) \ll K^2/D + K \log(2K).$$

Since $\kappa < 1/100$, all these quantities satisfy (34). The source was joined before these absolute estimates were taken. $\square$

13. PROOF OF THE SAMPLING LIMIT

Proof of Theorem 1.1. Take $H = \lfloor p^{11/14} \rfloor$ and $K_0 = p^{1/28}$. For $K > K_0$, combine (27), (23) and Theorem 12.1 to get

$$|\mathcal{B}_r(K; H)| \ll L_r^2 K^{-\kappa} \ell^3.$$

The complete source sum satisfies $\sum_r L_r^2 \leq N^2 \sum_{r \geq 1} r^{-2} \ll p^2$. The dyadic sum is geometric, so

$$\sum_r \sum_{K > K_0} |\mathcal{B}_r(K; H)| \ll p^{2-\kappa/28} \ell^3.$$

Its constant may depend on the fixed number κ and is absolute. For $K \leq K_0$, Proposition 9.1 instead gives

$$\sum_r \sum_{K \leq K_0} |\mathcal{B}_r(K; H)| \ll p^{5/3} H^{1/3} K_0 \ell^4 \leq p^{55/28} \ell^4,$$

using $\sum_r L_r \leq N H_N$ and $\sum_{K \leq K_0} K \ll K_0$. Thus the exact reconstruction (21) yields

$$|\mathcal{T}(H)| \ll p^{2-1/458696} \ell^3 + p^{55/28} \ell^4.$$

Insert this into (18). More explicitly,

$$\mathcal{S}_p = 1 + O(p^{-1/458696} \ell^3 + p^{-1/28} \ell^4 + p^{-1/14} \ell^3), \quad (36)$$

which implies (1). Proposition 2.1 then gives (2). All estimates have included every original source index and every admissible prefix depth. $\square$

14. THREE-CHANNEL SAMPLING CORRECTIONS

The difference between the two energy readings also has a finite three-channel interpretation. Keep $N = p - 1$ and $q = p^2$, and form the frequency Grams of the same sawtooths,

$$H_{rs}^d = \frac{4}{q} \sum_{a=0}^{q-1} \psi(ra/q) \psi(sa/q), \quad H_{rs}^c = 4 \int_0^1 \psi(rx) \psi(sx) dx = \frac{(r, s)^2}{3rs}.$$

Their total sums are E_p/q and $\mathcal{J}(N)$. To retain the diagonal and the two orderings of an off-diagonal pair, let Q be cyclic shift on $\mathbb{R}^3$ and define, for a real symmetric matrix H ,

$$\mathfrak{C}(H) = \text{tr}(H)I + \left(\sum_{r < s} H_{rs} \right) Q + \left(\sum_{r > s} H_{rs} \right) Q^{-1}.$$

This aggregation retains three sums from the frequency Gram. Its constant channel reads the total energy; the other two channels compare the diagonal with the off-diagonal interaction. Write $P_0 = \mathbf{1}\mathbf{1}^\top/3$ for the constant-line projection in $\mathbb{R}^3$, and $P_\perp = I - P_0$ for the centered-plane projection.

Corollary 14.1 (Cubic sampling correction). *For every odd prime,*

$$\mathfrak{C}(H^d) - \mathfrak{C}(H^c) = -\mathcal{S}_p P_0 + \left(\frac{\mathcal{S}_p}{2} - \frac{3N}{2q} + \frac{N}{q^2} \right) P_\perp. \quad (37)$$

Consequently, with $\eta = 1/458696$,

$$\|\mathfrak{C}(H^d) - \mathfrak{C}(H^c) + P_0 - \tfrac{1}{2}P_\perp\|_{\text{op}} \ll p^{-\eta} \log^4(2p).$$

The transverse eigenvalue of $\mathfrak{C}(H^d)$ is

$$\frac{N - \mathcal{J}(N)}{2} + \frac{1}{2} + O(p^{-\eta} \log^4(2p)).$$

Proof. Put $D(H) = \mathbf{1}^\top H \mathbf{1}$ and $\delta(H) = \text{tr}(H)$. Symmetry makes the upper and lower sums equal. Since $Q + Q^{-1} = 3P_0 - I$,

$$\mathfrak{C}(H) = D(H)P_0 + \frac{3\delta(H) - D(H)}{2} P_\perp.$$

Each $r \leq N$ is coprime to q , so its samples permute the grid. Summing the square of $2\psi(a/q)$ gives

$$\delta(H^d) = \frac{N(q-1)(q-2)}{3q^2}, \quad \delta(H^c) = \frac{N}{3}.$$

Subtract the two decompositions and use $D(H^c) - D(H^d) = \mathcal{S}_p$ to obtain (37). The operator-norm error is at most $|\mathcal{S}_p - 1| + 3N/(2q)$. Theorem 1.1 proves the stated limit and rate. Substituting $D(H^c) = \mathcal{J}(N)$ gives the transverse formula. $\square$

The half-unit corrections are measured against the exact continuous response. The scalar sampling limit determines all three corrections in this aggregation. The capacity deficit $N - \mathcal{J}(N)$ remains part of the transverse response.

15. A UNIT AT THE COMMON RESET

Correcting a single sample is enough to make the discrete and continuous readings agree asymptotically. To see this, change the squared profile only at the origin, setting $g_N(0) = N^2$ and $g_N(x) = B_N(x)^2$ otherwise. Its integral is unchanged and its grid average increases by N^2/p^2 . Therefore

$$\mathcal{S}_p = \frac{N^2}{p^2} + \left(\int_0^1 g_N(x) dx - \frac{1}{p^2} \sum_{a=0}^{p^2-1} g_N(a/p^2) \right).$$

The expression in parentheses tends to zero by Theorem 1.1. Equation (5) identifies it as $4\mathcal{A}_p/p^2 - NH_N/(3p^3)$. Thus the signed interior estimate has a direct meaning for the table. After correcting its common reset, the entire grid average approaches the continuous reading.

The correction also fixes the relation to the continuous capacity deficit. Exactly,

$$p - \frac{E_p}{p^2} = [N - \mathcal{J}(N)] + 1 + \mathcal{S}_p = [N - \mathcal{J}(N)] + 2 + o(1).$$

One unit comes from $p - N = 1$ and the other from the sampling limit. The continuous deficit $N - \mathcal{J}(N)$ carries its own finer asymptotics. Determining a third constant in the full energy expansion requires those terms as well; the remainder in the known capacity expansion still grows [7].

The next question is how quickly $\mathcal{S}_p$ approaches one. The numerical values in the secondary-term calculation [7] motivate a stronger rate.

Conjecture 15.1 (Square-root sampling rate). *For every $\varepsilon > 0$,*

$$\mathcal{S}_p = 1 + O_\varepsilon(p^{-1/2+\varepsilon})$$

as $p \rightarrow \infty$ through odd primes.

This bound allows fluctuations on either side of one. The origin contribution N^2/p^2 differs from one by $O(p^{-1})$, so the proposed square-root scale concerns the combined signed interior discrepancy. It asks for stronger cancellation in the same source $\mathcal{A}_p$, with all its weights and cutoff terms retained.

The cubic law and its secondary term describe the growth of the carry energy. The sampling limit identifies the persistent difference between the finite table and its continuous reading. An integral ignores the value assigned at a single point. The finite table records it.

The value one comes from the prescribed joint growth of the clocks and the grid. There are $N = p - 1$ clocks and $q = p^2$ samples, so the common reset contributes N^2/q . Its coherent square grows at the rate needed to balance the diminishing weight of a single sample. With the interior discrepancy controlled, this boundary contribution survives on its own. One sample accounts for the limiting difference of the entire table.

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